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FREE AND FORCED OSCILLATIONS IN A CLASS OF PIECEWISE-LINEAR DYNAMIC SYSTEMS

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ABSTRACT

A study is made of the free and forced oscillations in dynamic systems with hysteresis, on the basis of a piecewise-linear, non-linear model proposed by Reid. The existence, uniqueness, boundedness and periodicity of the solutions for a single degree of freedom system are established under appropriate conditions using topological methods and Brouwer's fixed-point theorem. Exact periodic solutions of a specified symmetry class are obtained and their stability is also examined. Approximate solutions have been derived by the Krylov-Bogoliubov-Van der Pol method and comparison is made with the exact solutions.

For dynamic systems with several degrees of freedom, consisting of "Reid oscillators", exact periodic solutions are derived under certain restricted forms of "modal excitation" and the stability of the periodic solutions has been studied. For a slightly more general form of sinusoidal excitation, a simple way of obtaining approximate solutions by "apparent superposition" has been indicated. Examples are presented on the exact and approximate periodic solutions in a dynamic system with two degrees of freedom.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The present investigation deals with the free and forced oscillations of a dynamic system with "linear hysteretic damping". The system restoring force is essentially derived from a linear spring; however, it is modified by a small additional contribution from a "hardening" or "softening" linear spring, depending upon whether loading or unloading takes place. The restoring force versus displacement characteristic is shown in Figure 1b, indicating clearly the dependence on velocity also. Alternately, the additional restoring force or the deviation from the linear characteristic may be considered as a damping force in phase with the velocity, but proportional to the magnitude of the displacement. As shown in Appendix I, the energy loss per cycle due to hysteresis, sustained by the system under sinusoidal vibrations, is proportional to the square of the amplitude, but independent of the frequency of oscillation. This fact enables one to adopt the system as a model to describe the behavior of hysteretic materials.

The concept of hysteretic damping

In 1927, Kimball and Lowell[1] discovered that many engineering materials exhibit a type of internal damping in which the energy loss per cycle is proportional to the square of the strain amplitude, but independent of the frequency at which the sinusoidal strain is applied. Wegel and Walther[2] confirmed the dependence on the square of the strain amplitude, but their observations indicated a weak dependence on frequency also. Recently, Lazan[3] has shown that below the fatigue limit for materials, the hysteresis loss is proportional to the square of the strain amplitude, but is essentially independent of frequency over a wide range of frequencies.

Closely following the discovery on the behavior of hysteretic materials by Kimball and Lowell, there evolved the concept of "linear hysteretic damping" in general engineering practice. Caughey[4] has remarked on the extensive application of this concept to air-craft flutter problems and in vibration theory. The papers by Duncan and Lyon[5], Theodoresen and Garrick[6] and the text by Scanlan and Rosenbaum[7] give a good account of the work carried out on air-craft flutter analysis.

In the field of vibrations, numerous papers have appeared on the subject of "linear hysteretic damping". The first among the recent ones was written by Mindlin[8] in 1948; then followed the papers by Soroka[9], Myklestad[10], Bishop[11], Reid[12], Fraeijs de Veubeke[13], Knopoff and MacDonald[14], Lancaster[15] and Caughey[4]. With the exception of Reid and Knopoff, all the papers listed above deal with linear models of "linear hysteretic damping". The various linear models and the basic non-linear model are briefly described in Appendix 1. Reid seems to be the first to have proposed this non-linear model, although in his paper he apparently failed to realize its essential non-linear character. For convenience, the non-linear dynamic system describing Reid's model, shall at times be referred to as the "Reid Oscillator" in the present work.

Piecewise-linear, non-linear systems

In the course of analysis, an autonomous or non-autonomous second order ordinary differential equation of the piecewise-linear, non-linear class is encountered. Loud[16] has discussed the advantages of piecewise-linear, non-linear models of physical systems over descriptions with non-linearities of cubic and other higher degree odd-polynomials, especially when large amplitudes are involved or no small parameters are inherent in the system. For a dynamic system with two-different spring constants depending upon whether the magnitude of the displacement is greater or less than unity, he has demonstrated the manifestation of the jump-phenomenon and of the existence of asymmetric periodic solutions even in systems possessing symmetry.

Fleishman[17] has analysed a certain relay control system of the on-off type; deriving periodic solutions, ultra- and sub-harmonics he has demonstrated the non-linear character of the problem. More strikingly, he has established the validity of the principle of convex superposition to obtain the response for certain types of inputs, to the special case of a "non-linear" system, he has considered.

By means of a convergent sequence of Fourier series expansions, Maezawa[18] has solved for a class of periodic solutions of a piecewise-linear conservative system, with special reference to the performance of ultrasonic machining devices. A certain type of rock drilling operation has been formulated in terms of a slightly modified version of the Reid oscillator by Fu[19], who has examined the stability of the periodic solution to the problem.

Scope of the present work

The objective of the present study is to examine the existence of the periodic solutions of a specified symmetry class, and their uniqueness and stability as applicable to a Reid oscillator. Exact and approximate techniques shall be used to construct the periodic solution with the desired symmetry. For dynamic systems with several degrees of freedom, exact periodic solutions have been obtained, for certain special "modal" forms of sinusoidal excitation; the stability analysis is presented also, in these cases. A simple way of obtaining approximate solutions by "apparent superposition" has been established for multi-degree of freedom systems, when the restriction on the modal form of the excitation is relaxed. In this context, it is anticipated that this attempt shall be a further step in the study of piecewise-linear, non-linear dynamic systems with definite engineering applications.

CHAPTER II

THE FREE VIBRATIONS OF A REID OSICLLATOR

Consider the free oscillations of a mass M attached to a "Reid spring" as shown in Figure 1. The equation of motion is given by

$$M\ddot{x} + kx\left\{1+g\ \mathrm{sgn}(x\dot{x})\right\}=0 \tag{2.1a}$$

$$x(0)=a\ ,\quad \dot{x}(0)=0 \tag{2.1b}$$

where k is the spring constant, g is the "non-linearity parameter" and $\dot{x}$ denotes dx/dt. The "signum" function is defined as

$$\mathrm{sgn}(\theta)=\begin{cases} +1 & \text{for } \theta>0 \\ 0 & \theta=0 \\ -1 & \theta<0 \end{cases} \tag{2.2}$$

It shall be assumed throughout $0<g\ll 1$. Let

$$\omega^2=\frac{k}{M}\ ,\quad \tau=\omega t \tag{2.3}$$

Substituting (2.3) into (2.1),

$$x'' + \left\{1+g\ \mathrm{sgn}(xx')\right\}x=0 \tag{2.4a}$$

$$x(0)=a\ ,\quad x'(0)=0 \tag{2.4b}$$

where

$$x'=\frac{dx}{d\tau}\ .$$

Since the restoring force in (2.4a) is bounded, piecewise continuous and has only finite, discrete discontinuities, the conditions of the Cauchy-Lipschitz theorem (see Struble[20], page 43) are satisfied

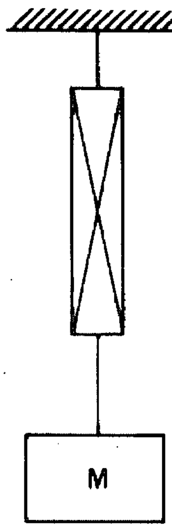


Figure 1a
The Reid Oscillator

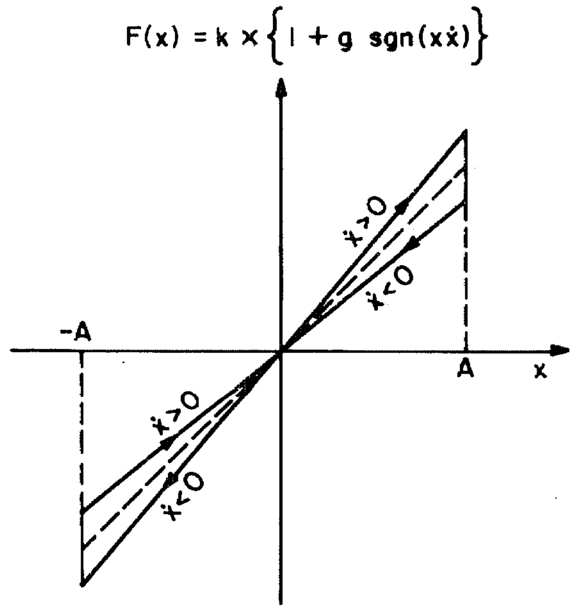


Figure 1b
Restoring Force vs. Displacement Characteristic of the Reid Oscillator

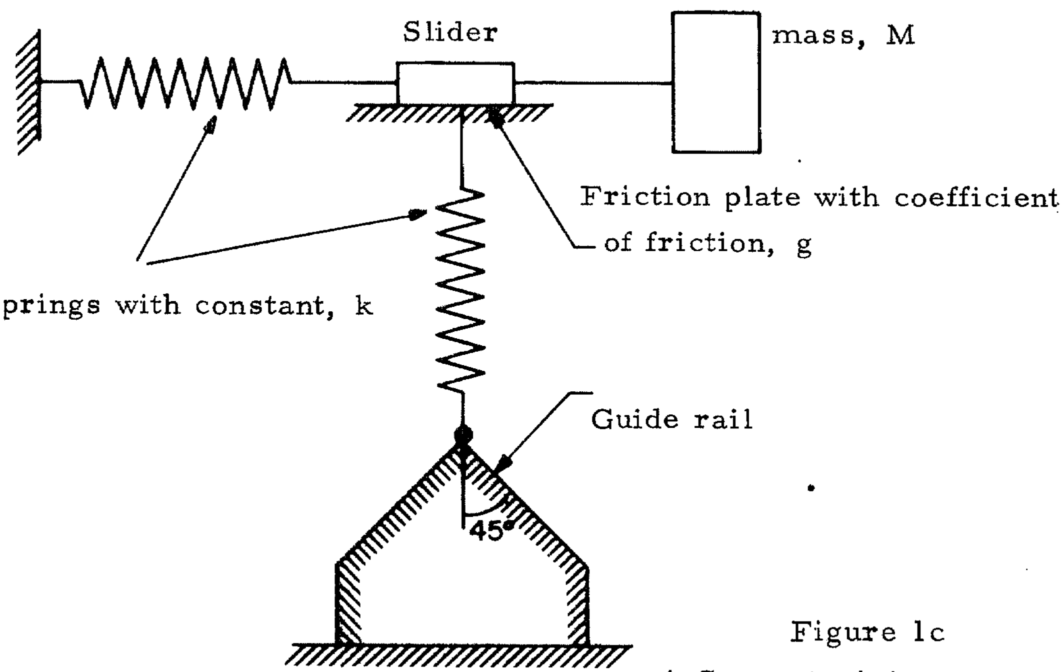


Figure 1c
A Conceptual Arrangement of the Reid Oscillator

and the initial value problem in (2.4) has a unique solution with x and x' continuous in τ and the initial values. More details concerning this question are presented in Chapter III, while dealing with the corresponding non-autonomous differential equation.

It will also be noticed that if in (2.4a), x and x' are replaced by -x and $-x'$ respectively, the equation remains unaltered so that it is sufficient to consider semi-trajectories only. For $0<\tau<\tau_1$, let $x'<0$, $x>0$ and $x(\tau_1)=0$. From (2.4a),

$$x'' + (1-g)\,x = 0 \tag{2.5}$$

Solving (2.5) for the initial values in (2.4b),

$$x = a\cos\sqrt{1-g}\;\tau \tag{2.6}$$

$$\tau_1 = \frac{\pi}{2\sqrt{1-g}} \tag{2.7}$$

and

$$x'(\tau_1) = -a\sqrt{1-g} \tag{2.8}$$

For $\tau_1<\tau<\tau_2$, let $x'<0$, $x<0$ and $x'(\tau_2)=0$. Then (2.4a) becomes,

$$x'' + (1+g)\,x = 0 \tag{2.9a}$$

with

$$x(\tau_1)=0\ ,\quad x'(\tau_1) = -a\sqrt{1-g} \tag{2.9b}$$

Hence, for

$$\tau_1<\tau<\tau_2,\ x = -\frac{a\sqrt{1-g}}{\sqrt{1+g}}\ \sin\sqrt{1+g}\ (\tau-\tau_1) \tag{2.10}$$

Since

$$x'(\tau_2)=0\ ,\quad \tau_2 = \tau_1 + \frac{\pi}{2\sqrt{1+g}} \tag{2.11}$$

$$= \frac{\pi}{2} \left\{ \frac{1}{\sqrt{1-g}} + \frac{1}{\sqrt{1+g}} \right\} \qquad (2.11a)$$

And

$$x(\tau_2) = -a\sqrt{\frac{1-g}{1+g}} \qquad (2.12)$$

The symmetry property of (2.4a) implies that $\tau_2 = \tau_d/2$, where τ_d is the period of damped oscillations,

$$\tau_d = \pi \frac{\{\sqrt{1+g} + \sqrt{1-g}\}}{\sqrt{1-g^2}} \qquad (2.13)$$

The frequency of damped oscillations is given by,

$$\omega_d = \frac{2\pi}{\tau_d} = \frac{2\sqrt{1-g^2}}{\sqrt{1+g} + \sqrt{1-g}} \qquad (2.14)$$

For

$$g << 1 \;, \quad \omega_d \simeq \left(1 - \frac{3}{8} g^2\right) \qquad (2.15)$$

Again from the symmetry of (2.4a),

$$x(\tau_d) = -x(\tau_d/2)\,\frac{\sqrt{1-g}}{\sqrt{1+g}} = a\left(\frac{1-g}{1+g}\right) \qquad (2.16)$$

Therefore,

$$\frac{x(\tau_d)}{x(0)} = \frac{1-g}{1+g} \qquad (2.17)$$

The logarithmic decrement δ is therefore given by,

$$\delta = \ell n\left(\frac{x(0)}{x(\tau_d)}\right) = \ell n\,\frac{1+g}{1-g} \qquad (2.18)$$

For

$$g << 1 \;, \quad \delta \simeq 2g \qquad (2.19)$$

For a viscously damped system with damping coefficient β,

$$\overline{\omega}_d \simeq 1 - \frac{\beta^2}{8} \tag{2.20}$$

and

$$\overline{\delta} \simeq \pi\beta$$

Selecting $\beta = 2g/\pi$, would help to make the two decrements δ and $\overline{\delta}$ equal; however,

$$\overline{\omega}_d \simeq 1 - \frac{4}{\pi^2} \frac{g^2}{8} > \omega_d$$

If quantities of $0(g^2)$ are neglected, then

$$\overline{\omega}_d = \omega_d = 1 \quad ; \quad \overline{\delta} = \delta$$

Thus, if $g << 1$, the system with the "Reid spring" behaves in almost the same way, as a viscously damped system with a damping ratio $\zeta = \frac{\beta}{2} = \frac{g}{\pi}$, at least as far as the free oscillations are concerned.

CHAPTER III

FORCED OSCILLATIONS IN A SINGLE DEGREE OF FREEDOM SYSTEM

III.1 The Existence of A Unique Periodic Solution

Consider the forced oscillations of a mass M attached to a "Reid spring" and acted upon by a force, F(t). The equation of motion is given by,

$$M\ddot{x} + kx\left\{1+g\ \mathrm{sgn}\,(x\dot{x})\right\} = F(t) \tag{3.1a}$$

$$x(0) = a\ ,\ \ \dot{x}(0) = b \tag{3.1b}$$

Let

$$\omega_n^2 = \frac{k}{M}\ ,\quad \tau = \omega_n t \text{ and } \frac{F(t)}{k} = f(\tau).$$

Then (3.1) becomes,

$$x'' + x\,(1+g\ \mathrm{sgn}\ xx') = f(\tau) \tag{3.2a}$$

$$x(0) = a\ ,\ \ x'(0) = b/\omega_n \tag{3.2b}$$

<u>Theorem 3.1</u>

If in (3.2a), $f(\tau)$ is piecewise continuous and bounded, then the initial value problem in (3.2) possesses a unique solution, with $x(\tau)$ and $x'(\tau)$ continuous in τ and the initial values in (3.2b).

<u>Proof</u>

Rewriting (3.2) in the matrix-vector notation,

$$\frac{d\underline{x}}{d\tau} = \begin{bmatrix} 0 & 1 \\ -(1+g\ \mathrm{sgn}\ xx') & 0 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ f(\tau) \end{bmatrix} = \underline{h}(\underline{x},\tau) \tag{3.3a}$$

$$\underline{x}(0) = \begin{bmatrix} a \\ b/\omega_n \end{bmatrix} \tag{3.3b}$$

where

$$\underline{x} = \begin{bmatrix} x \\ x' \end{bmatrix}.$$

If $f(\tau)$ is piecewise continuous and bounded, then in any finite region of the phase-plane (x-x' plane),

$$\| \underline{h}(\underline{x}, \tau) \| \leq K \tag{3.4}$$

for any suitable vector norm and an appropriate constant, K. Moreover, for any pair of vectors $\underline{x}$ and $\underline{y}$ lying entirely in the upper-half or in the lower-half of the phase-plane

$$\| \underline{h}(\underline{x}, \tau) - \underline{h}(\underline{y}, \tau) \| \leq (1+g) \| \underline{x} - \underline{y} \| \tag{3.5}$$

Thus, the conditions of the Cauchy-Lipschitz theorem for a non-autonomous system (see Struble[20], page 56) are satisfied so that a unique solution to (3.2) exists, the trajectory being continuous in τ, a and b and can be continued indefinitely or until $\tau = \tau^+$ is reached, where $x'(\tau^+) = 0$. But obviously, any segment of the x-axis cannot be a part of the solution trajectories of (3.2a); also for a non-constant function $f(\tau)$, a point solution cannot exist. Therefore, after $x'(\tau^+) = 0$ has been reached, the solution curve must necessarily enter a region ($x' \gtrless 0$), where the Lipschitz condition (3.5) holds, besides (3.4), so that it can be continued further under the same arguments as before.

Continuity of the solution in the initial values

The continuous dependence of the solution on the initial data follows directly on the application of the Lipschitz condition (3.5), wherever it is satisfied (see Struble[20], page 48). The only difficulty arises when the two initial vectors $\underline{c}_1$ and $\underline{c}_2$ are such that both the initial displacements are positive or both negative, but the initial velocities are necessarily of opposite signs. Then the Lipschitz condition (3.5) does not hold necessarily, until the velocity of one solution changes in sign.

Since the forward and backward problems are well-posed, it would be sufficient to consider the following case, to establish the continuous dependence of the solution on the initial values.

Let $x(\tau)$ and $y(\tau)$ be two solutions of (3.2a) with initial conditions $x(0)$, $x'(0)$, $y(0)$ and $y'(0) > 0$. Also, for $0 \le \tau \le \tau^{**}$, let $x(\tau)$, $y(\tau) > 0$. However, let

$$y'(\tau) > 0 \text{ for } 0 \le \tau < \tau^{*}$$

$$y'(\tau^{*}) = 0$$

and

$$y'(\tau) < 0 \text{ for } \tau^{*} < \tau < \tau^{**}.$$

Similarly, let $x'(\tau) > 0$, $0 \le \tau < \tau^{**}$, with $x'(\tau^{**}) = 0$. It will be noticed that for $\tau^{*} < \tau < \tau^{**}$, the Lipschitz condition in (3.5) is not generally valid.

For

$$0 < \tau < \tau^{*}, \quad x'' + (1+g)x = f(\tau), \quad \underline{x}(0) = \underline{c} \tag{3.6}$$

$$y'' + (1+g)y = f(\tau), \quad y(0) = \underline{c}^{*} \tag{3.7}$$

Let

$$\underline{x} - \underline{y} = \underline{z}, \quad \underline{z}(0) = \underline{c} - \underline{c}^{*} = \underline{d}^{*}.$$

Then from (3.6) and (3.7),

$$z'' + (1+g)z = 0 \tag{3.8a}$$

$$\underline{z}(0) = \underline{d}^{*} \tag{3.8b}$$

Solving (3.8),

$$\underline{z} = \begin{bmatrix} \cos\sqrt{1+g}\ \tau & \dfrac{1}{\sqrt{1+g}}\ \sin\sqrt{1+g}\ \tau \\ -\sqrt{1+g}\ \sin\sqrt{1+g}\ \tau & \cos\sqrt{1+g}\ \tau \end{bmatrix} \underline{d}^* \tag{3.9}$$

so that,

$$\|\underline{z}(\tau)\| \leq (2+g)\ \|\underline{d}^*\| = (2+g)\ \|\underline{c}-\underline{c}^*\| \quad \text{for} \quad 0 \leq \tau \leq \tau^* \tag{3.10}$$

But since $y'(\tau^*) = 0$, and $|x' - y'| \leq \|\underline{x}-\underline{y}\|$,

$$|x'(\tau^*)| \leq (2+g)\ \|\underline{c}-\underline{c}^*\| \tag{3.11}$$

Similarly,

$$|x(\tau^*) - y(\tau^*)| \leq (2+g)\ \|\underline{c}-\underline{c}^*\| \tag{3.12}$$

For

$$\tau^* < \tau < \tau^{**}, \quad x'' + (1+g)x = f(\tau) \tag{3.13a}$$

$$y'' + (1-g)y = f(\tau) \tag{3.13b}$$

so that

$$(x-y)'' + (1+g)\ (x-y) = -2gy \tag{3.14}$$

with

$$\underline{x}(\tau^*) - \underline{y}(\tau^*) = \underline{e}^* \tag{3.15}$$

being the initial value vector. As before, let $\underline{x}-\underline{y} = \underline{z}$. Then,

$$\underline{z} = Z\,e^* + \int_{\tau^*}^{\tau} Z(\tau - s) \begin{bmatrix} 0 \\ -2g\ y(s) \end{bmatrix} ds \tag{3.16}$$

where $Z(\tau)$ is the principal matrix solution for the homogeneous problem in (3.14) and $Z(0) = I$, the unit diagonal matrix. Obviously, from (3.14),

$$Z(\tau) = \begin{bmatrix} \cos\sqrt{1+g}\ \tau & \frac{1}{\sqrt{1+g}} \sin\sqrt{1+g}\ \tau \\ -\sqrt{1+g}\ \sin\sqrt{1+g}\ \tau & \cos\sqrt{1+g}\ \tau \end{bmatrix} \quad (3.17)$$

In particular,

$$\underline{z}(\tau^{**}) = Z(\tau^{**})\underline{e}^* + \int_{\tau^*}^{\tau^{**}} Z(\tau^{**}-s) \begin{bmatrix} 0 \\ -2g\ \ y(s) \end{bmatrix} ds$$

Therefore,

$$\|\underline{z}(\tau^{**})\| \leq (2+g)\ \|\underline{e}^*\| + 2g(2+g)\int_{\tau^*}^{\tau^{**}} |y(s)|\, ds \quad (3.18)$$

However, as already discussed under the proof on the existence of solutions, for $\tau^* \leq \tau \leq \tau^{**}$, $|y(\tau)|$ is necessarily bounded. Then, from (3.18),

$$\|\underline{z}(\tau^{**})\| \leq (2+g)\ \|\underline{e}^*\| + 2g(2+g)\ |y_{max}|\ (\tau^{**}-\tau^*) \quad (3.19)$$

where

$$|y_{max}| = \max_{\tau^* \leq s \leq \tau^{**}} |y(s)|$$

Moreover, since $x'(\tau)$ is continuous in the closed interval $\tau^* \leq \tau \leq \tau^{**}$, and $x''(\tau)$ is defined as in (3.13a) in the open interval $\tau^* < \tau < \tau^{**}$, from the mean value theorem,

$$x'(\tau^{**}) = x'(\tau^*) + (\tau^{**} - \tau^*)\ x''(\xi) \quad (3.20)$$

where

$$\tau^* \leq \xi \leq \tau^{**}$$

In particular, since $x''(\tau)$ cannot be identically zero,

$$x'(\tau^{**}) = 0 \text{ implies } \tau^{**} - \tau^{*} = -\frac{x'(\tau^{*})}{x''(\xi)} \tag{3.21}$$

Substituting (3.11) into (3.21) gives,

$$(\tau^{**} - \tau^{*}) \le \frac{(2+g)\,\|\underline{c}-\underline{c}^{*}\|}{|x''(\xi)|} \tag{3.22}$$

From (3.19), (3.22), (3.15) and (3.12),

$$\|\underline{x}(\tau^{**}) - \underline{y}(\tau^{**})\| \le (2+g)^2 \; \|\underline{c}-\underline{c}^{*}\| + 2g(2+g)\,|y_{max}|\,\frac{(2+g)}{|x''(\xi)|}\,\|\underline{c}-\underline{c}^{*}\|$$

$$\le (2+g)^2 \left\{ 1 + 2g\,\frac{|y_{max}|}{|x''(\xi)|} \right\} \; \|\underline{c}-\underline{c}^{*}\| \tag{3.23}$$

Hence, as $\|\underline{c}-\underline{c}^{*}\| \to 0$, so also,

$$\|\underline{x}(\tau^{**}) - \underline{y}(\tau^{**})\| \to 0 \tag{3.24}$$

Thus the solution depends continuously on the initial values.

<u>Corollary 3.1</u>

It may be noted that, if in (3.2), $f(\tau)$ is replaced by $\beta\, f(\tau)$ and x by $\beta x = y$, then the equation

$$y'' + \{1 + g\,\mathrm{sgn}(yy')\}y = \beta\, f(\tau) \tag{3.25a}$$

$$y(0) = \beta\, a \;\; , \;\; y'(0) = \beta\, b/\omega_n \tag{3.25b}$$

has the solution

$$y(\tau) = \beta\, x(\tau) \tag{3.26}$$

for any scalar constant β.

<u>Theorem 3.2</u>

If in the equation

$$x'' + (1 + g \operatorname{sgn} xx')\, x = f(\tau) \tag{3.27}$$

$f(\tau)$ is piecewise continuous and bounded, then all solutions of (3.27) are ultimately bounded with,

$$|x| \le \frac{2(f_0 + \epsilon)}{g} \tag{3.28}$$

where $f(0) = \sup_{\tau} |f(\tau)|$ and $\epsilon > 0$.

Before proceeding to the proof of Theorem 3.2, it would be convenient to derive certain results to be subsequently used in comparison.

Lemma 3.1

The equation

$$x'' + (1 + g \operatorname{sgn} xx')x = f_1 \operatorname{sgn} x' \tag{3.29}$$

possesses a unique, stable limit cycle, $\Gamma(f_1)$, where f_1 is a constant.

Proof

Consider the initial value problem,

$$x'' + (1 + g \operatorname{sgn} xx')\, x = f_1 \operatorname{sgn} x' \tag{3.30a}$$

$$x(0) = a \;;\; x'(0) = 0 \tag{3.30b}$$

By treating f_1 as the constant value of a dummy dependent variable $f^*(\tau)$, a simple extension (see Struble[20], page 62) of Theorem 3.1 guarantees the existence of a unique solution to the initial value problem in (3.30) with x and x' continuous in τ, a and the parameter f_1.

Moreover, substituting in (3.30a),

$$x = -y \;,\; x' = -y' \tag{3.31}$$

it is seen that the equation remains unaltered so that it is sufficient to consider only semi-trajectories of the motion.

Let

$$x(\tau) > 0 \ , \ \ x'(\tau) < 0 \ , \ \ 0 < \tau < \tau_1$$

with (3.32)

$$x'(0) = 0 \ \text{and} \ x(\tau_1) = 0$$

From (3.30a)

$$x'' + (1-g)\,x + f_1 = 0 \tag{3.33}$$

Multiplying (3.33) by x' and integrating, taking into consideration the initial values in (3.30b)

$$\frac{x'^2}{2} + \frac{x^2}{2}(1-g) + f_1 x = \frac{1}{2} a^2 (1-g) + f_1 a \tag{3.34}$$

Since

$$x(\tau_1) = 0 \ , \ \ x'(\tau_1) = -\sqrt{a^2(1-g) + 2 f_1 a} \tag{3.35}$$

For $\tau_1 < \tau < \tau_2$, let $x(\tau) < 0$, $x'(\tau) < 0$ with $x(\tau_1) = 0$, $x'(\tau_2) = 0$.

During this interval

$$x'' + (1+g)\,x + f_1 = 0 \tag{3.36}$$

so that

$$\frac{1}{2} x'^2 + \frac{1}{2}(1+g)\,x^2 + f_1 x = \text{constant} \tag{3.37}$$

Taking into account the initial values in (3.35) and from the continuity of $x(\tau)$ and $x'(\tau)$,

$$\frac{1}{2} x'^2 + \frac{1}{2}(1+g)\,x^2 + f_1 x = \frac{1}{2} a^2 (1-g) + f_1 a \tag{3.38}$$

Since

$$x'(\tau_2) = 0 \quad , \quad \frac{(1+g)}{2}x^2(\tau_2) + f_1 x(\tau_2) = \frac{1}{2}a^2(1-g) + f_1 a \tag{3.39}$$

For a limit cycle,

$$x(\tau_2) = -x(0) = -a \tag{3.40}$$

From (3.39) and (3.40)

$$ag\left(a - \frac{2f_1}{g}\right) = 0 \tag{3.41}$$

or, for a non-trivial solution, a must be equal to a^*, where

$$a^* = \frac{2f_1}{g} \tag{3.42}$$

Thus, if $x(0) = a^* = 2f_1/g$, $x'(0) = 0$, then the trajectory of (3.30a) passes through

$$x(\tau_1) = 0 \quad , \quad x'(\tau_1) = -\sqrt{a^{*2}(1-g) + 2f_1 a^*} = -v^* = -\frac{2f_1}{g} \tag{3.43}$$

and

$$x(\tau_2) = -a^* = -\frac{2f_1}{g} \quad , \quad x'(\tau_2) = 0 \tag{3.44}$$

From the symmetry property of (3.30a), it follows

$$x(\tau_2+\tau_1) = 0 \quad , \quad x'(\tau_2+\tau_1) = \sqrt{a^{*2}(1-g) + 2f_1 a^*} = v^* = \frac{2f_1}{g} \tag{3.45}$$

and

$$x(2\tau_2) = a^* \quad , \quad x'(2\tau_2) = 0 \tag{3.46}$$

Since the solution of the initial value problem is unique, there exists a unique limit cycle $\Gamma(f_1)$, consisting of elliptical arcs, passing through the points, $(a^*, 0)$, $(0, -v^*)$, $(-a^*, 0)$ and $(0, v^*)$, where

$$v^* = a^* = \frac{2f_1}{g} .$$

(3.34) and (3.37) are the equations of the elliptic arcs in the appropriate quadrants in the lower half of the phase-plane.

The closed bounded set of points in the x-x′ plane, enclosed by the limit cycle $\Gamma(f_1)$ shall be denoted by the compact set, $\Omega_o(f_1)$. Since a^* given in (3.42), is proportional to f_1 increasing f_1 increases the size of the limit cycle. Furthermore, the uniqueness of the solution, rules out any two limit cycles intersecting each other or having points in common. Therefore, a) $\Omega_o(f_1) \subset \Omega_o(f_2)$ for $f_2 > f_1$ and b) the limit cycles $\Gamma(f_1)$ form a set of nested curves with no points in common.

Stability of the limit cycle

To examine the stability of the limit cycle $\Gamma(f_1)$, consider a function G(x) defined by,

$$G(x) = x^2 + \frac{2f_1 x}{(1+g)} - \frac{a^2(1-g) + 2f_1 a}{(1+g)} \tag{3.47}$$

For $0<g<1$, G(x) has the following properties:

a) $G(x) \le 0$ consistent with (3.38) to give real values to the velocity

b) $G(0) < 0$

c) $G'(x) = 2x + \frac{2f_1}{1+g}$, so that

$$G'(x) \gtreqless 0 \text{ accordingly as } x \gtreqless -\frac{f_1}{1+g}$$

d) $G\left(-\frac{f_1}{1+g}\right) < G(0) < 0$, directly from property (c).

$$G(x) < G(0) \ , \quad \text{for } -\frac{f_1}{1+g} \le x < 0$$

$$G(x) > G\left(-\frac{f_1}{1+g}\right) \text{ for } x < -\frac{f_1}{1+g}$$

e) $G(-a) = \dfrac{2ag\left(a - \dfrac{2f_1}{g}\right)}{(1+g)}$

Therefore

$$G(-a) > 0 \text{ if } a > a^* = \frac{2f_1}{g} > \frac{f_1}{1+g}$$

$$G(-a) < 0 \text{ if } a < a^*$$

f) From properties (d) and (e), given above, it is seen that $G(x) = 0$ implies that,

i) $x = x(\tau_2) > -a$, if $a > a^* = \dfrac{2f_1}{g}$ and

ii) $x = x(\tau_2) < -a$, if $a < a^*$.

From the symmetry property of (3.30) and the observation on $x(\tau_2)$ in (f) above, it is obvious that if the motion is started at a point outside the limit cycle $\Gamma(f_1)$, the trajectories will spiral inwards, towards the limit cycle; if on the other hand, the initial values lie inside the limit cycle, the trajectories will spiral outwards towards the limit cycle. Thus the limit cycle $\Gamma(f_1)$ is stable.

Lemma 3.2

All solutions of Equation (3.2), starting outside the limit cycle $\Gamma(f_1)$ are ultimately bounded in $\Omega_o(f_1)$, where $f_1 = f_o + \epsilon$, $f_o = \sup_\tau |f(\tau)|$, $\epsilon > 0$.

Proof

From (3.2),

$$\frac{dx'}{dx} = \frac{-x(1+g \operatorname{sgn} xx') + f(\tau)}{x'} \tag{3.48}$$

Therefore,

$$\left.\frac{dx'}{dx}\right|_{x'=0} = \pm \infty \ , \quad \text{provided } |x| > \frac{f_o}{(1-g)} \tag{3.49}$$

Thus, for sufficiently large values of $|x|$, the axis contains no arc of the trajectory. Let $x(\tau_o) = a$, $x'(\tau_o) = 0$ and for $\tau_o < \tau < \tau_1$, $x'(\tau) < 0$ $x(\tau) > 0$ with $x(\tau_1) = 0$. Suppose

$$V = \frac{1}{2}x'^2 + \frac{1}{2}(1-g)x^2 + f_1 x \tag{3.50}$$

Evidently $V \geq 0$ where equality holds if and if only $x = x' = 0$. From (3.50),

$$V' = x'x'' + (1-g)xx' + f_1 x' \tag{3.51}$$

Evaluating V' along a trajectory of (3.2),

$$V'_{(3.2)} = (f_1 - f(\tau))x'(\tau) \tag{3.52}$$

Since $\sup_{\tau} |f(\tau)| = f_o < f_1$ and for $\tau_o < \tau < \tau_1$, $x'(\tau) < 0$,

$$V'_{(3.2)} \leq -(f_1 - f_o)|x'| = -\epsilon|x'| < 0 \tag{3.53}$$

So along a trajectory of (3.2), V decreases and for $\tau_o < \tau < \tau_1$,

$$V(\tau) - V(\tau_o) \leq \epsilon(x-a) < 0 \tag{3.54}$$

Evaluating V' along a trajectory of the comparison equation in (3.30), it is seen that,

$$V'_{(3.30)} = 0 \tag{3.55}$$

so that along a trajectory of (3.30), V remains constant.

For different values of V, the Equation (3.50) represents a one parameter family of ellipses, all with their centers at the same point, $\left(-\frac{f_1}{1-g}, 0\right)$ in the phase-plane, and the length of the major axis,

$$p = \frac{2}{\sqrt{1-g}} \sqrt{2V + \frac{f_1^2}{1+g}} \tag{3.56}$$

and the length of the minor axis,

$$q = 2\sqrt{2V + \frac{f_1^2}{1+g}} \tag{3.57}$$

Since p and q both increase with increasing values of V, the ellipses form a set of nested curves, with no points in common.

For $\tau_o < \tau < \tau_1$, along the trajectories of Equation (3.30) V remains constant, while V decreases along the trajectories of (3.2). Thus on examining these trajectories, both passing through (a, 0) at $\tau = \tau_o$, it is seen that the integral curves of (3.2) lie closer to the origin than those of (3.30), for $\tau_o < \tau < \tau_1$, in view of (3.56) and (3.57). Similar arguments, in all the other three quadrants of the phase-plane, yield the same result.

Moreover, from Lemma 3.1, the trajectories of Equation (3.30) starting outside the limit cycle $\Gamma(f_1)$, ($a > a^*$ at $\tau = \tau_o$), spiral inwards to the limit cycle; so it follows that the trajectories of (3.2), also spiral inwards. If the motion were to be started from points lying on $\Gamma(f_1)$, still for a non-constant function $f(\tau)$ with $\sup_\tau |f(\tau)| = f_o < f_1$, $V'_{(3.2)} < 0$ over intervals during which $x' \neq 0$; for $x' = 0$ and $|x| > f_o/1-g$, the x-axis contains no arc of the trajectory. Therefore, it follows that the trajectories of the Equation (3.2), starting outside $\Gamma(f_1)$, must eventually intersect $\Gamma(f_1)$ and enter the interior of $\Omega_o(f_1)$.

Lemma 3.3

All solutions of the Equation (3.2) starting inside the set $\Omega_o(f_o)$ must remain inside $\Omega_o(f_1)$, where $f_o = \sup_{\tau} |f(\tau)|$ and $f_1 = f_o + \varepsilon$, $\varepsilon > 0$.

Proof

If in Equation (3.50) under Lemma 3.2, f_1 were to be replaced by f_o and the arguments carried through, it would be concluded that trajectories starting in the annular set $(\Omega_o(f_1) - \Omega_o(f_o))$, never intersect $\Gamma(f_1)$ and indeed tend to the limit cycle $\Gamma(f_o)$ for Equation (3.30) or enter the points interior to $\Gamma(f_o)$ for Equation (3.2), as time τ increases without bound.

Consider a trajectory which lies in the interior of the set $\Omega_o(f\ )$ at $\tau = \tau_o$. Let it be assumed that at a later time $\tau = \tau_2$, the trajectory lies outside the limit cycle $\Gamma(f_1)$. Since $\Omega_o(f_o)$ is a subset of $\Omega_o(f_1)$ with no boundary points in common, there must exist a time τ_1, $\tau_o < \tau_1 < \tau_2$, such that this trajectory lies in the annulus $(\Omega_o(f_1) - \Omega_o(f_o))$. But it has already been shown that a trajectory starting in the above annulus can never intersect the limit cycle $\Gamma(f_1)$; therefore, it is impossible for the trajectory of (3.2) to leave the interior of $\Omega_o(f_1)$.

Proof of Theorem 3.2

It is desired to show that, given $f(\tau)$ is piecewise continuous and bounded, then

a) all solutions of the equation

$$x'' + (1 + g \operatorname{sgn} xx')\, x = f(\tau)$$

are ultimately bounded in the set $\Omega_o(f_1)$ and

b) $\max_{\tau \to \infty} |x(\tau)| \leq \frac{2f_1}{g}$, where $f_1 = f_o + \epsilon$,

$$f_o = \sup_{\tau} |f(\tau)| \ , \quad \epsilon > 0.$$

The result in (a) above follows directly from Lemmas 3.2 and 3.3; the result in (b) follows from Lemma 3.1, where it has been shown that on $\Gamma(f_1)$, $|x| \leq 2f_1/g$ and the fact that the trajectories of (3.2) are ultimately bounded in $\Omega_o(f_1)$ as just mentioned in (a) above.

Theorem 3.3

If in Equation (3.2), $f(\tau)$ is periodic of period T, besides being bounded and piecewise continuous, then there exists at least one periodic solution of (3.2), of period T.

Proof

By Theorem 3.2, all solutions of the Equation (3.2) are ultimately bounded in the set $\Omega_o(f_1)$ and, in particular, solutions starting in $\Omega_o(f_1)$ must remain inside $\Omega_o(f_1)$ for all τ.

It has already been shown in Theorem 3.1, that the solutions to the initial value problem in (3.2), namely $x(\tau)$ and $x'(\tau)$, exist, are unique and continuous in τ as well as the initial values. Hence there exists a continuous mapping M(T), which maps points $[x(\tau), x'(\tau)]$ in $\Omega_o(f_1)$ into points $[x(\tau+T), x'(\tau+T)]$, which are also in $\Omega_o(f_1)$. Therefore, by the Brouwer's fixed point theorem, (see Saaty[21], page 42) there must exist at least one fixed point $[x_o, x'_o]$ in $\Omega_o(f_1)$. Thus

$$[x_o(\tau+T), \ x'_o(\tau+T)] = M(T)[x_o(\tau), \ x'_o(\tau)] = [x_o(\tau), \ x'_o(\tau)] \qquad (3.58)$$

Similarly,

$$[x_o(\tau + 2T),\ x_o'(\tau + 2T)] = M(T)[x_o(\tau + T),\ x_o'(\tau + T)]$$

$$= M(T)[x_o(\tau),\ x_o'(\tau)] = [x_o(\tau),\ x_o'(\tau)] \quad (3.59)$$

So also

$$[x_o(\tau + nT),\ x_o'(\tau + nT)] = M(T)[x_o(\tau + (n-1)T),\ x_o'(\tau + (n-1)T)]$$

$$= \cdots = [x_o'(\tau),\ x_o'(\tau)] \quad (3.60)$$

Hence there exists at least one periodic solution of (3.2) of period T.

III.2 Construction of An Exact Periodic Solution

In this section an exact periodic solution shall be derived to satisfy the differential equation,

$$x'' + (1+g \text{ sgn } xx')x = P \sin(\omega\tau + \varphi) \tag{3.61}$$

and also have the property,

$$x(\tau + \pi/\omega) = -x(\tau) \tag{3.62}$$

Evidently from (3.61) and (3.62), the excitation and the solution both have the same period $(2\pi/\omega)$. During each half-cycle, $x(\tau)$ shall monotonically increase or decrease and accordingly, $x'(\tau)$ shall stay positive or negative during an entire half-cycle except that at the beginning and at the end of the half cycle, $x'(\tau) = 0$. These stipulations considerably simplify the algebra involved in determining the periodic solution.

At $\tau = 0$, let $x(0) = -A$, $x'(0) = 0$ and that for $0 < \tau < \tau_1 = \frac{\alpha\pi}{\omega}$, $0 < \alpha < 1$, $x(\tau) < 0$, $x'(\tau) > 0$ with $x(\tau_1) = 0$. Then from Equation (3.61),

$$x'' + (1-g)x = P \sin(\omega\tau + \varphi) \tag{3.63a}$$

$$x(0) = -A, \quad x'(0) = 0 \tag{3.63b}$$

(3.63) can easily be solved, provided $\omega^2 \neq (1-g)$, to get

$$x(\tau) = \frac{-P\omega\cos\varphi}{\sqrt{1-g}\,(1-g-\omega^2)} \sin\sqrt{1-g}\,\tau - \left\{A + \frac{P\sin\varphi}{1-g-\omega^2}\right\} \cos\sqrt{1-g}\,\tau + \frac{P}{1-g-\omega^2}\sin(\omega\tau+\varphi) \tag{3.64}$$

$$x'(\tau) = -\frac{P\omega\cos\varphi}{1-g-\omega^2}\cos\sqrt{1-g}\,\tau + \sqrt{1-g}\left\{A + \frac{P\sin\varphi}{1-g-\omega^2}\right\}\sin\sqrt{1-g}\,\tau + \frac{P\omega}{1-g-\omega^2}\cos(\omega\tau+\varphi) \tag{3.65}$$

At $\tau = \tau_1 = \alpha\pi/\omega$,

$x(\alpha\pi/\omega) = 0$

$$= -\frac{P\omega\cos\varphi}{\sqrt{1-g}\,(1-g-\omega^2)}\sin\sqrt{1-g}\,\frac{\alpha\pi}{\omega} - \left\{A + \frac{P\sin\varphi}{1-g-\omega^2}\right\}\cos\sqrt{1-g}\,\frac{\alpha\pi}{\omega}$$

$$+ \frac{P}{1-g-\omega^2}\sin(\alpha\pi+\varphi) \qquad (3.66)$$

$$x'(\alpha\pi/\omega) = -\frac{P\omega\cos\varphi}{1-g-\omega^2}\cos\sqrt{1-g}\,\frac{\alpha\pi}{\omega} + \sqrt{1-g}\left\{A + \frac{P\sin\varphi}{1-g-\omega^2}\right\}\sin\sqrt{1-g}\,\frac{\alpha\pi}{\omega}$$

$$+ \frac{P\omega}{1-g-\omega^2}\cos(\alpha\pi+\varphi) \qquad (3.67)$$

Suppose that for $\alpha\pi/\omega = \tau_1 < \tau < \tau_2 = \pi/\omega$, $x(\tau) > 0$, $x'(\tau) > 0$ with the condition that $x'(\pi/\omega) = 0$. Then (3.61) becomes,

$$x'' + (1+g)x = P\sin(\omega\tau+\varphi)\ ;\ \frac{\alpha\pi}{\omega} < \tau < \frac{\pi}{\omega}\,. \qquad (3.68a)$$

$$x\left(\frac{\alpha\pi}{\omega}\right) = 0\ ;\ x'\left(\frac{\alpha\pi}{\omega}\right) = b \qquad (3.68b)$$

where b is given by Equation (3.67). The solution to (3.68) is easily obtained by elementary methods (provided $\omega^2 \neq 1+g$).

$$x(\tau) = \left\{\frac{b}{\sqrt{1+g}} - \frac{P\omega\cos(\varphi+\alpha\pi)}{\sqrt{1+g}\,(1+g-\omega^2)}\right\}\sin\sqrt{1+g}\left(\tau - \frac{\alpha\pi}{\omega}\right)$$

$$-\frac{P\sin(\varphi+\alpha\pi)}{1+g-\omega^2}\cos\sqrt{1+g}\left(\tau - \frac{\alpha\pi}{\omega}\right) + \frac{P\sin(\omega\tau+\varphi)}{1+g-\omega^2} \qquad (3.69)$$

$$x'(\tau) = \left\{b - \frac{P\omega\cos(\varphi+\alpha\pi)}{(1+g-\omega^2)}\right\}\cos\sqrt{1+g}\left(\tau - \frac{\alpha\pi}{\omega}\right)$$

$$+\sqrt{1+g}\,\frac{P\sin(\varphi+\alpha\pi)}{1+g-\omega^2}\sin\sqrt{1+g}\left(\tau - \frac{\alpha\pi}{\omega}\right) + \frac{P\omega\cos(\omega\tau+\varphi)}{1+g-\omega^2} \qquad (3.70)$$

The closure condition for a periodic solution, of period $T = \frac{2\pi}{\omega}$, having the symmetry of (3.62) is that

$$x(\pi/\omega) = -x(0) = A \ , \quad x'(\pi/\omega) = -x'(0) = 0 \ .$$

Thus,

$$A = \left\{ \frac{b}{\sqrt{1+g}} - \frac{P\omega \cos(\varphi + \alpha\pi)}{\sqrt{1+g}\,(1+g-\omega^2)} \right\} \sin\sqrt{1+g}\ \frac{(1-\alpha)\pi}{\omega}$$

$$- \frac{P \sin(\varphi+\alpha\pi)}{1+g-\omega^2} \cos\sqrt{1+g}\ \frac{(1-\alpha)\pi}{\omega} + \frac{P\sin(\pi+\varphi)}{1+g-\omega^2} \tag{3.71}$$

$$0 = \left\{ b - \frac{P\omega\cos(\varphi+\alpha\pi)}{1+g-\omega^2} \right\} \cos\sqrt{1+g}\ \frac{(1-\alpha)\pi}{\omega}$$

$$+ \sqrt{1+g}\ \frac{P\sin(\varphi+\alpha\pi)}{1+g-\omega^2} \sin\sqrt{1+g}\ \frac{(1-\alpha)\pi}{\omega} + \frac{P\omega\cos(\pi+\varphi)}{1+g-\omega^2} \tag{3.72}$$

Eliminating A and b from Equations (3.66), (3.67), (3.71) and (3.72),

$$c\cos\varphi + d\sin\varphi = 0 \tag{3.73}$$

$$e\cos\varphi + f\sin\varphi = 0 \tag{3.74}$$

Therefore

$$cf - ed = 0 \tag{3.75}$$

$$\tan\varphi = -\frac{c}{d} = -\frac{e}{f} \tag{3.76}$$

$$c = J + \frac{R\cos\eta}{1 - \sqrt{\frac{1-g}{1+g}}\sin\beta\sin\eta} \qquad d = K + \frac{Q\cos\eta}{1 - \sqrt{\frac{1-g}{1+g}}\sin\beta\sin\eta}$$

$$e = L - \frac{R\cos\beta\sin\eta}{\sqrt{1+g} - \sqrt{1-g}\,\sin\beta\sin\eta} \qquad f = M - \frac{Q\cos\beta\sin\eta}{\sqrt{1+g} - \sqrt{1-g}\,\sin\beta\sin\eta}$$

$$J = -D\omega \cos\alpha\pi \cos\eta + D\sqrt{1+g} \sin\alpha\pi \sin\eta - D\omega$$

$$K = D\omega \sin\alpha\pi \cos\eta + D\sqrt{1+g} \cos\alpha\pi \sin\eta$$

$$L = -A_0 \sin\delta + B\sin\alpha\pi + C\cos\alpha\pi \sin\eta \cos\beta + D\sin\alpha\pi \cos\eta \cos\beta$$

$$M = -B\cos\beta + B\cos\alpha\pi - C\sin\alpha\pi \sin\eta \cos\beta + D\cos\alpha\pi \cos\eta \cos\beta + D\cos\beta$$

$$Q = \sqrt{1-g}\, B\sin\beta - B\omega \sin\alpha\pi + \sqrt{1-g}\, C\sin\alpha\pi \sin\eta \sin\beta - \sqrt{1-g}\, D\cos\alpha\pi \cos\eta \sin\beta - \sqrt{1-g}\, D\sin\beta$$

$$R = -B\omega\cos\beta + B\omega\cos\alpha\pi - \sqrt{1-g}\, C\cos\alpha\pi \sin\eta \sin\beta - D\sqrt{1-g} \sin\alpha\pi \cos\eta \sin\beta$$

$$A_0 = \frac{P\omega}{\sqrt{1-g}\,(1-g-\omega^2)} \quad ; \quad C = \frac{P\omega}{\sqrt{1+g}\,(1+g-\omega^2)}$$

$$B = \frac{P}{1-g-\omega^2} \quad ; \quad D = \frac{P}{1+g-\omega^2}$$

$$\beta = \sqrt{1-g}\,\frac{\alpha\pi}{\omega} \quad ; \quad \eta = \sqrt{1+g}\,\frac{(1-\alpha)\pi}{\omega} \tag{3.77}$$

The solution technique used was to give values of g and ω and to solve equation (3.75) numerically for α; the "phase" φ was then obtained from (3.76) and the "amplitude" A, from Equation (3.66). The values so obtained for α, A and φ must be substituted back in Equations (3.64), (3.65), (3.69) and (3.70) and it must be verified that $\mathrm{sgn}(xx') = -1$, $0 < \tau < \alpha\pi/\omega$ and $\mathrm{sgn}(xx') = +1$ in the interval $\alpha\pi/\omega < \tau < \pi/\omega$ to avoid extraneous roots obtained in solving the transcendental Equation (3.75).

The numerical results are shown in Figures 2 and 3 for three values of $g = 0.05$, 0.1 and 0.2 and for ω in the range $0.5 < \omega < 3.0$.

It is to be recalled that ω as in Figures 2 and 3 actually refers to the frequency ratio ω_e/ω_n, where ω_e is the excitation frequency and ω_n is the natural frequency. In the calculations, P was taken to be 1. It may be noticed that the phase φ, as defined above, goes through zero at resonance; this is to draw attention to the fact that since the exact response is not harmonic, φ is not the phase shift between the fundamental Fourier component of the solution and the input, nor is A the amplitude in the usual sense.

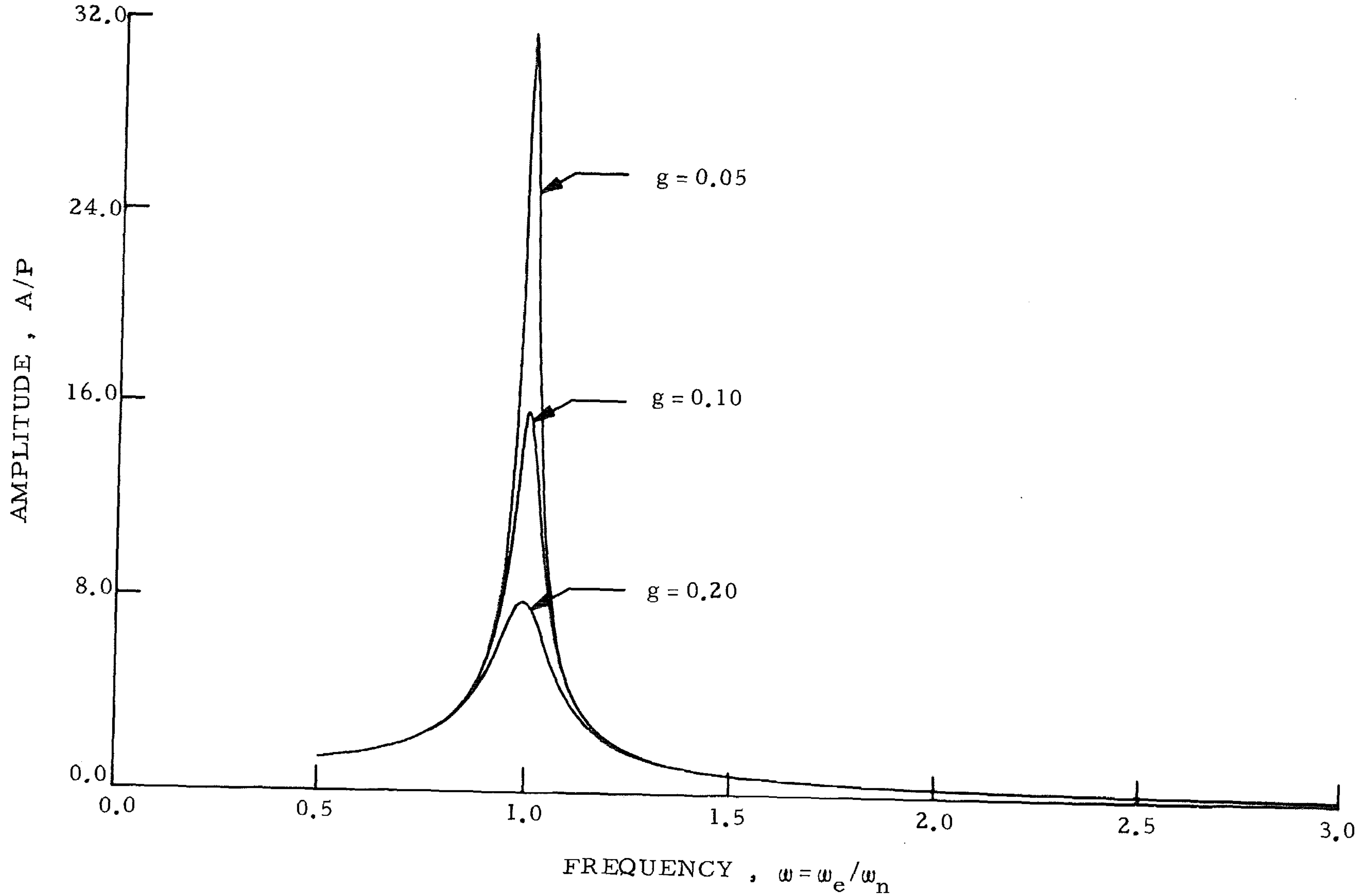


Figure 2a. Steady State Response of the Reid Oscillator

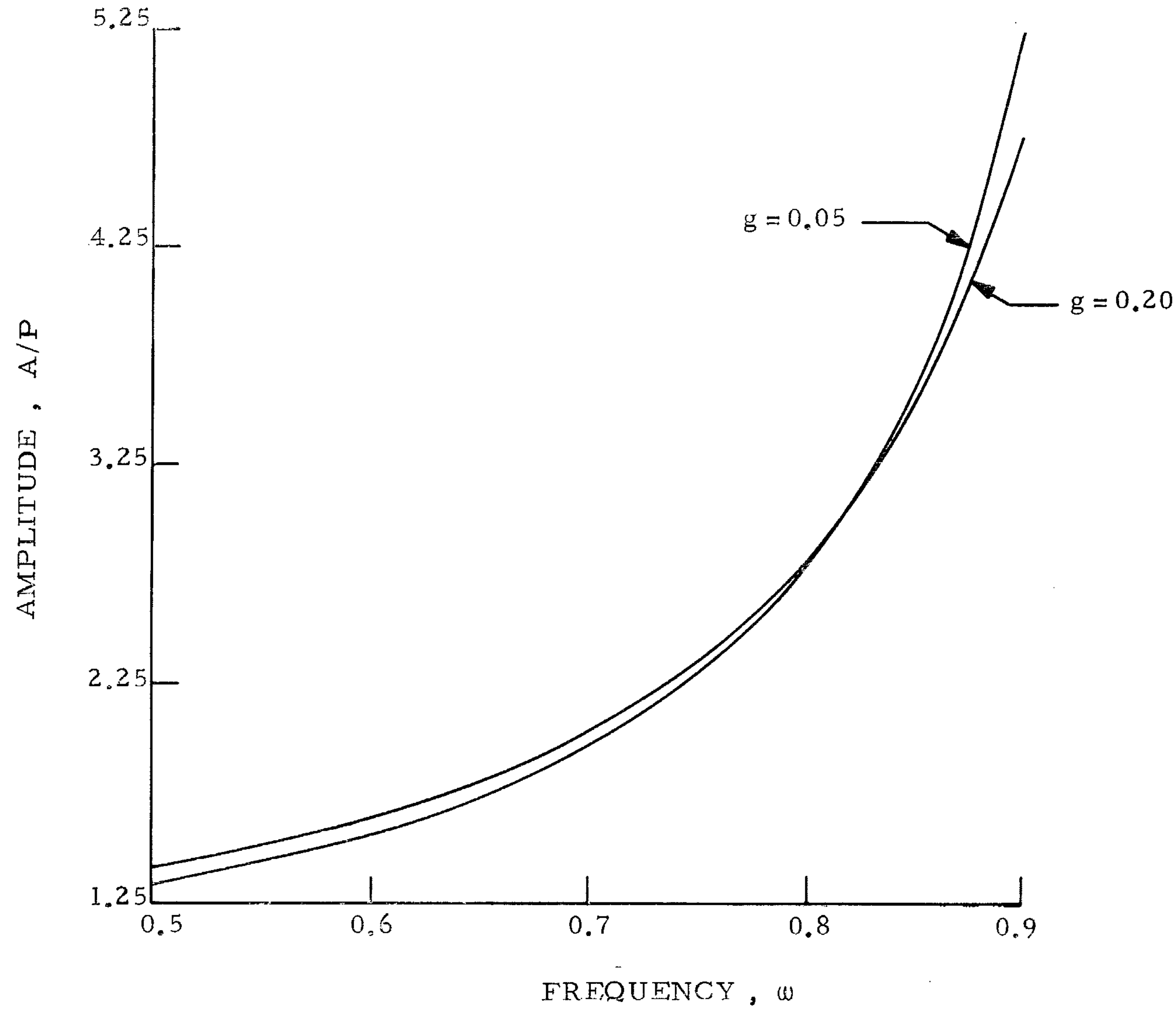
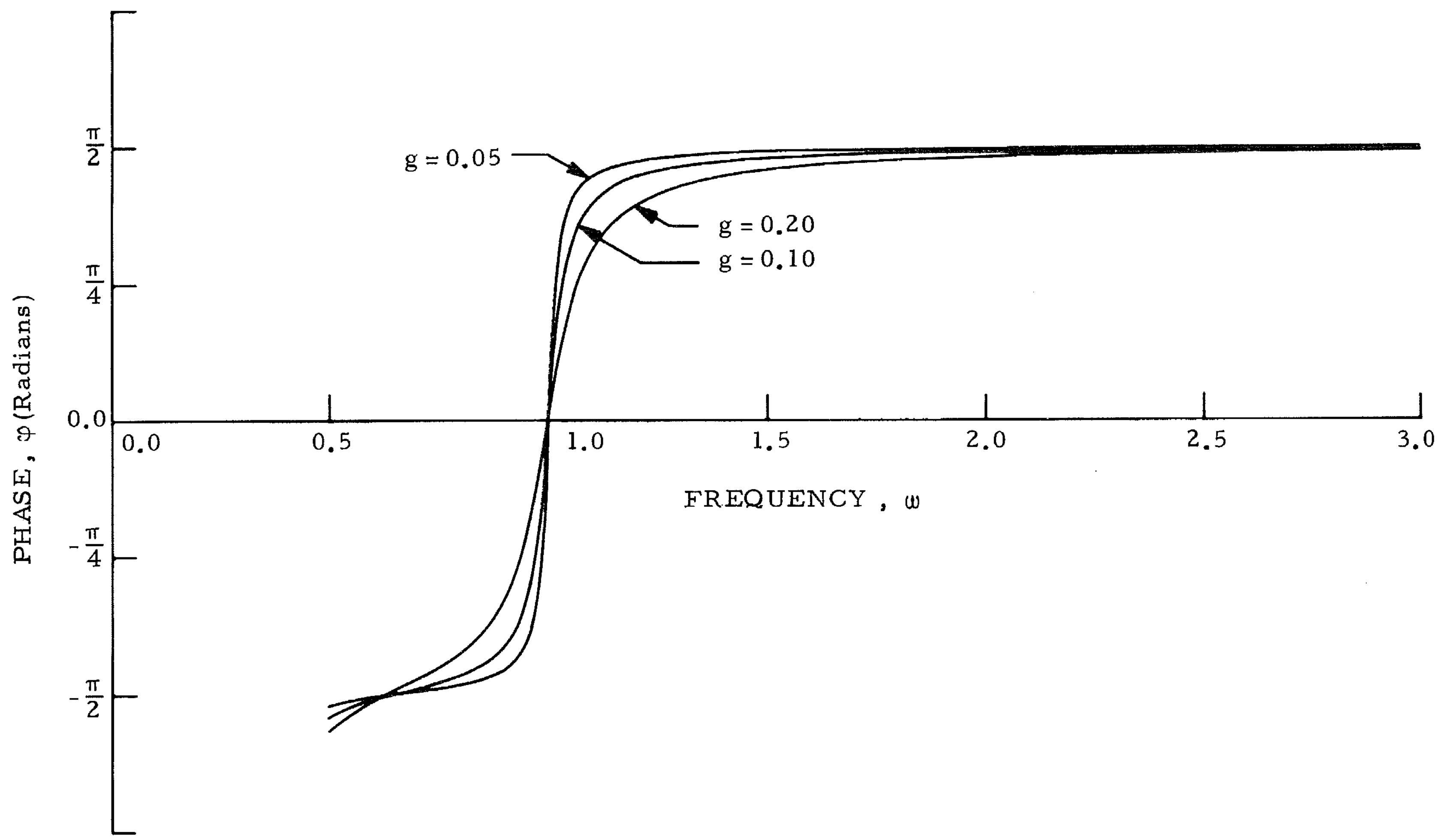


Figure 2b. Details of Low Frequency Response



g = 0.05
g = 0.20
g = 0.10
PHASE, φ (Radians)
$\frac{\pi}{2}$
$\frac{\pi}{4}$
0.0
$-\frac{\pi}{4}$
$-\frac{\pi}{2}$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
FREQUENCY , ω

Figure 3.

III.3 Stability Analysis of the Periodic Solution

The existence theorem, Theorem 3.3, guaranteed the existence of at least one periodic solution of period $T = 2\pi/\omega$, the method of construction given in (III.2), yielded a single periodic solution with the symmetry in (3.62). However, this solution is not guaranteed to be stable. The question of stability must be examined separately. Since Equation (3.61) is of second order, it is acceptable to examine the behavior of any two quantities such as displacement and velocity or amplitude and phase to determine the stability of the periodic solution. For the dynamic system with single degree of freedom, it is found convenient to study the amplitude and phase as defined in Section III.2.

Let the differential equation of motion (3.61) be expressed in the form

$$\underline{z}' = \underline{F}(\underline{z}, \tau) \tag{3.78}$$

and let a particular solution of (3.78) be

$$\underline{z} = \underline{s}(\tau) \tag{3.79}$$

If this solution is perturbed slightly, so that

$$\underline{z}_p = \underline{s}(\tau) + \underline{\xi}(\tau) \tag{3.80}$$

then the original solution in (3.79) is said to be asymptotically stable in the sense of Liapunov, if for any $\delta > 0$, there exists an $\epsilon > 0$ such that for $\|\underline{\xi}(0)\| \leq \epsilon$, $\|\underline{\xi}(\tau)\| < \delta \;\; \forall\, \tau > 0$ and $\mathcal{L}t_{\tau \to \infty} \|\underline{\xi}(\tau)\| = 0$. In the usual cases, the time-behavior of $\underline{\xi}(\tau)$ is furnished by the so-called variational equations

$$\underline{\xi}' = M\, \underline{\xi} \tag{3.81}$$

where

$$M_{ij} = \frac{\partial F_i(s, \tau)}{\partial s_j} \quad (i, j = 1, 2) \qquad (3.82)$$

Unfortunately, because of the sgn(xx') term in (3.61), the matrix M cannot be obtained by classical methods. However, by borrowing ideas from error propagation in difference equations, following the perturbation at the beginning of a cycle, the deviation from the periodic solution can be determined at the end of the cycle. Repeating this process over and again, we can follow the propagation (similar to change with time) of the initial perturbations from the periodic solution. The stability or instability of the solution is then determined by whether or not the deviations from the periodic solution decay or grow, as the number of cycles increases without bound (i.e. as $\tau \to \infty$).

In general the deviations in amplitude and phase are determined by non-linear difference equations. It may be shown, however (see for example Masri and Caughey[22]) that if the solutions of the linearized difference equations are asymptotically stable, then so also are the solutions of the non-linear difference equations, provided the initial deviations are sufficiently small.

Let $x_o(\tau)$ denote the periodic solution as determined in the previous section, with phase φ_o, of the differential equation

$$x'' + (1 + g \,\mathrm{sgn}\, xx')x = P \sin(\omega\tau + \varphi) \qquad (3.83)$$

Let the perturbation in the displacement x be ξ; suppose that at $\tau = \tau_o$, $x(\tau_o) = -A$, $x'(\tau_o) = 0$, then

$$\left.\begin{array}{ll} x = x_o + \xi_o = -A_o - \Delta A_o\ , & \varphi = \varphi_o + \Delta\varphi_o \\ x' = x'_o + \xi'_o = 0 & ,\ \ \xi_o = -\Delta A_o \end{array}\right\} \tag{3.84}$$

where $\Delta\varphi_o$ is the perturbation in φ, ΔA_o is the perturbation in A at $\tau = \tau_o$, x and x' denote the displacement and velocity in the perturbed state.

Suppose also that for $\tau_o < \tau < \tau_1$, $x' > 0$, $x(\tau_o) < 0$. Let f(x) be defined by

$$\begin{aligned} f(x) &= (1-g)\,x\ , \quad x<0 \\ & (1+g)\,x\ , \quad x>0 \end{aligned} \tag{3.85}$$

The equation of first variation is

$$\frac{d^2\xi}{d\tau^2} + f'(x_o)\xi = P\cos(\omega\tau + \varphi_o)\Delta\varphi \tag{3.86}$$

while

$$\frac{d^2 x_o}{d\tau^2} + f(x_o) = P\sin(\omega\tau + \varphi_o) \tag{3.87}$$

Multiply both sides of (3.87) by $\Delta\varphi_o/\omega$ and differentiate with respect to τ,

$$\frac{d^2}{d\tau^2}\left[\frac{x'_o\Delta\varphi_o}{\omega}\right] + f'(x_o)\left[\frac{x'_o\,\Delta\varphi_o}{\omega}\right] = P\cos(\omega\tau + \varphi_o)\Delta\varphi_o \tag{3.88}$$

Subtracting (3.88) from (3.86)

$$\frac{d^2}{d\tau^2}\left[\xi - \frac{x'_o\Delta\varphi_o}{\omega}\right] + f'(x_o)\left[\xi - \frac{x'_o\,\Delta\varphi_o}{\omega}\right] = 0 \tag{3.89}$$

Let

$$u(\tau) = \xi - x_o'(\tau)\,\frac{\Delta\varphi_o}{\omega} \tag{3.90}$$

Then

$$\frac{d^2u}{d\tau^2} + f'(x_o(\tau))\,u = 0 \tag{3.91}$$

This is a special case of Mathieu-Hill equation. Let u_1 and u_2 be two solutions of (3.91), where

$$\begin{aligned} u_1(\tau_o) &= 0 \quad ; \quad u_1'(\tau_o) = 1 \\ u_2(\tau_o) &= 1 \quad ; \quad u_2'(\tau_o) = 0 \end{aligned} \tag{3.92}$$

At

$$\tau = \tau_o\,,\quad u(\tau_o) = \xi(\tau_o) - \frac{x'(\tau_o)\Delta\varphi_o}{\omega} = -\Delta A_o \tag{3.93}$$

$$u'(\tau_o) = \xi'(\tau_o) - x''(\tau_o)\,\frac{\Delta\varphi_o}{\omega} = -x''(\tau_o)\,\frac{\Delta\varphi_o}{\omega} \tag{3.94}$$

Then

$$u(\tau) = -\Delta A_o\, u_2(\tau) - x_o''(\tau_o)\,\frac{\Delta\varphi_o}{\omega}\, u_1(\tau) \tag{3.95}$$

Suppose that at $\tau = \tau_o + \frac{\pi}{\omega} + \Delta\tau_1$, $x'(\tau) = x_o'(\tau) + \xi'(\tau) = 0$. Then,

$$\begin{aligned} \xi(\tau_o + \tfrac{\pi}{\omega} + \Delta\tau_1) &= u(\tau_o + \pi/\omega + \Delta\tau_1) + x_o'(\tau_o + \tfrac{\pi}{\omega} + \Delta\tau_1)\,\frac{\Delta\varphi_o}{\omega} \\ &= u(\tau_o + \pi/\omega) + u'(\tau_o + \tfrac{\pi}{\omega})\,\Delta\tau_1 \\ &\quad + x_o'(\tau + \tfrac{\pi}{\omega})\,\frac{\Delta\varphi_o}{\omega} + x_o''(\tau_o + \tfrac{\pi}{\omega})\,\frac{\Delta\varphi_o}{\omega}\,\Delta\tau_1 \\ &= u(\tau_o + \pi/\omega) + \text{quantities of second order in } \Delta\tau,\ \Delta\varphi_o,\ \Delta A\,. \end{aligned} \tag{3.96}$$

So at the end of the first half cycle,

$$\xi_1 = \xi(\tau_o + \pi/\omega + \Delta\tau_1) = \Delta A_1 = u(\tau_o + \pi/\omega)$$

$$= -\Delta A_o\, u_2(\tau_o + \frac{\pi}{\omega}) - x_o''(\tau_o)\, \frac{\Delta\varphi_o}{\omega}\, u_1(\tau_o + \frac{\pi}{\omega}) \tag{3.97}$$

From Equation (3.87),

$$x_o''(\tau_o) = P \sin\varphi_o + (1-g) A_o \tag{3.98}$$

$$x_o''(\tau_o + \frac{\pi}{\omega}) = -\left\{P \sin\varphi_o + (1+g) A_o\right\} \tag{3.99}$$

At $\tau = \tau_o + \frac{\pi}{\omega}$,

$$\xi'(\tau_o + \frac{\pi}{\omega}) = u'(\tau_o + \frac{\pi}{\omega}) + x_o''(\tau_o + \frac{\pi}{\omega})\, \frac{\Delta\varphi_o}{\omega}$$

$$= -\Delta A_o\, u_2'(\tau_o + \frac{\pi}{\omega}) - x_o''(\tau_o)\, \frac{\Delta\varphi_o}{\omega}\, u_1'(\tau_o + \frac{\pi}{\omega}) + x_o''(\tau_o + \frac{\pi}{\omega})\, \frac{\Delta\varphi_o}{\omega}$$

$$= -\Delta A_o\, u_2'(\tau_o + \frac{\pi}{\omega}) - \frac{\Delta\varphi_o}{\omega}\left[x_o''(\tau_o)\, u_1'(\tau_o + \frac{\pi}{\omega}) - x_o''(\tau_o + \frac{\pi}{\omega})\right] \tag{3.100}$$

At $\tau = \tau_o + \frac{\pi}{\omega} + \Delta\tau_1$,

$$x'(\tau) = 0 = x_o'(\tau_o + \frac{\pi}{\omega}) + \xi'(\tau_o + \frac{\pi}{\omega}) + \left[x''(\tau_o + \frac{\pi}{\omega}) + \xi''(\tau_o + \frac{\pi}{\omega})\right]\Delta\tau_1$$

$$= \xi'(\tau_o + \frac{\pi}{\omega}) + x''(\tau_o + \frac{\pi}{\omega})\Delta\tau_1 + \text{quantities of second order}$$

$$\text{in } \Delta\tau,\ \Delta\varphi_o,\ \Delta A\ . \tag{3.101}$$

Therefore,

$$\Delta\tau_1 = -\frac{\xi'(\tau_o + \frac{\pi}{\omega})}{x''(\tau_o + \frac{\pi}{\omega})} = \frac{\left\{\Delta A_o\, u_2'(\tau_o + \frac{\pi}{\omega}) + \frac{\Delta\varphi_o}{\omega}\, x_o''(\tau_o)\, u_1'(\tau_o + \frac{\pi}{\omega})\right\}}{x_o''(\tau_o + \frac{\pi}{\omega})} - \frac{\Delta\varphi_o}{\omega} \tag{3.102}$$

At the end of the half-cycle, $\Delta\varphi_1 = \Delta\varphi_o + \omega\Delta\tau_1$.

So

$$\Delta\varphi_1 = \frac{\omega\Delta A_o\, u_2'(\tau_o + \frac{\pi}{\omega})}{x_o''(\tau_o + \frac{\pi}{\omega})} + \Delta\varphi_o \frac{x_o''(\tau_o)}{x_o''(\tau_o + \frac{\pi}{\omega})} u_1'(\tau_o + \frac{\pi}{\omega}) \tag{3.103}$$

In matrix-vector notation,

$$\begin{bmatrix} \Delta A_1 \\ \Delta\varphi_1 \end{bmatrix} = \begin{bmatrix} -u_2(\tau_o + \frac{\pi}{\omega}) & -\dfrac{x_o''(\tau_o)\, u_1(\tau_o + \frac{\pi}{\omega})}{\omega} \\ \dfrac{\omega\, u_2'(\tau_o + \frac{\pi}{\omega})}{x_o''(\tau_o + \frac{\pi}{\omega})} & \dfrac{x_o''(\tau_o)}{x_o''(\tau_o + \frac{\pi}{\omega})} u_1'(\tau_o + \frac{\pi}{\omega}) \end{bmatrix} \begin{bmatrix} \Delta A_o \\ \Delta\varphi_o \end{bmatrix} \tag{3.104}$$

For the next half cycle, $x' < 0$, $x(\tau_o + \pi/\omega + \Delta\tau_1) = A_o + \Delta A_1$, $\varphi = \varphi_o + \Delta\varphi_1$. A similar analysis to that performed for the first half cycle shows that

$$\begin{bmatrix} \Delta A_2 \\ \Delta\varphi_2 \end{bmatrix} = M \begin{bmatrix} \Delta A_1 \\ \Delta\varphi_1 \end{bmatrix} \tag{3.105}$$

where

$$M = \begin{bmatrix} -u_2(\tau_o + \frac{\pi}{\omega}) & -\dfrac{x_o''(\tau_o)\, u_1(\tau_o + \frac{\pi}{\omega})}{\omega} \\ \dfrac{\omega\, u_2'(\tau_o + \frac{\pi}{\omega})}{x_o''(\tau_o + \frac{\pi}{\omega})} & \dfrac{x_o''(\tau_o)}{x_o''(\tau_o + \frac{\pi}{\omega})} u_1'(\tau_o + \frac{\pi}{\omega}) \end{bmatrix} \tag{3.106}$$

Thus,

$$\begin{bmatrix} \Delta A_2 \\ \Delta\varphi_2 \end{bmatrix} = M^2 \begin{bmatrix} \Delta A_o \\ \Delta\varphi_o \end{bmatrix} \tag{3.107}$$

In general, the deviations in the amplitude and phase at the end of the n^{th} cycle are given by

$$\begin{bmatrix} \Delta A_n \\ \Delta\varphi_n \end{bmatrix} = M^n \begin{bmatrix} \Delta A_o \\ \Delta\varphi_o \end{bmatrix} \tag{3.108}$$

A necessary and sufficient conditon that ΔA_n, $\Delta\varphi_n$ tend to zero as n tends to infinity is that the eigen values of M be less than unity in modulus. The eigen values of M are determined form,

$$\lambda^2 + \lambda\left[u_2(\tau_o + \tfrac{\pi}{\omega}) - \frac{x_o''(\tau_o)}{x_o''(\tau_o + \frac{\pi}{\omega})}\, u_1'(\tau_o + \tfrac{\pi}{\omega})\right] - \frac{x_o''(\tau_o)}{x_o''(\tau_o + \frac{\pi}{\omega})}\left[W(\tau_o + \tfrac{\pi}{\omega})\right] = 0 \tag{3.109}$$

where $W(\tau_o + \frac{\pi}{\omega}) = W(\tau_o) = 1$ is the Wronskian of the solutions of (3.91).

The solutions u_1, u_2 of (3.91) are easily obtained by elementary method.

$$\left.\begin{aligned}
u_1(\tau_o + \tfrac{\pi}{\omega}) &= \frac{1}{\sqrt{1+g}} \cos\beta \, \sin\eta + \frac{1}{\sqrt{1-g}} \sin\beta \, \cos\eta \\
u_1'(\tau_o + \tfrac{\pi}{\omega}) &= \cos\beta \, \cos\eta - \sqrt{\frac{1+g}{1-g}} \sin\beta \, \sin\eta \\
u_2(\tau_o + \tfrac{\pi}{\omega}) &= \cos\beta \, \cos\eta - \sqrt{\frac{1-g}{1+g}} \sin\beta \, \sin\eta \\
u_2'(\tau_o + \tfrac{\pi}{\omega}) &= -\sqrt{1-g} \, \sin\beta \, \cos\eta - \sqrt{1+g} \, \cos\beta \, \sin\eta
\end{aligned}\right\} \tag{3.110}$$

where

$$\beta = \sqrt{1-g}\ \frac{\alpha\pi}{\omega}, \quad \eta = \sqrt{1+g}\ \frac{(1-\alpha)\pi}{\omega}$$

$0<\alpha<1$, such that $x_o(\tau_o+\alpha\pi/\omega) = 0$ as determined by solving Equation (3.75). From Equations (3.98), (3.99), (3.109) and (3.110),

$$\lambda^2 + \frac{2\lambda}{(1+g)A_o + P\sin\varphi_o}\Bigg[(A_o + P\sin\varphi_o)\cos\beta\cos\eta - \frac{P\sin\varphi_o + (1-g^2)A_o}{\sqrt{1-g^2}}\cdot\sin\beta\ \sin\eta\Bigg] + \frac{(1-g)A_o + P\sin\varphi_o}{(1+g)A_o + P\sin\varphi_o} = 0 \qquad (3.111)$$

With the values of A_o, φ_o and α for the periodic solutions obtained in the earlier section, it was found that the periodic solutions with the symmetry of Equation (3.62) were stable for the values of the parameter $g = 0.05$, 0.1 and 0.2 for $\omega > 0.5$.

III. 4 A Different Approach to the Stability Problem

The stability problem, discussed in the preceding section, can be formulated in yet another way from first principles, leading however to exactly the same results as in Equation (3.111). This latter approach is found to be more convenient, for instance, in determining the stability of the periodic response of dynamic systems with several degrees of freedom. So presently, the equivalence of the two methods shall be demonstrated explicitly.

Analysis from first principles

Once again, let $x_o(\tau)$ represent the exact periodic solution constructed in Section III. 2. From earlier discussion it follows

$$x_o'' + (1 + g \operatorname{sgn} x_o x_o')x_o = P \sin(\omega\tau + \varphi_o) \qquad (3.112a)$$

$$x_o(0) = -A_o \quad ; \quad x_o'(0) = 0 \qquad (3.112b)$$

Moreover for

$$\left.\begin{array}{llll} 0 < \tau < \frac{\pi}{\omega} & , & x_o'(\tau) > 0 \text{ with } x_o'(\frac{\pi}{\omega}) = 0\,; & \\ 0 < \tau < \frac{\alpha\pi}{\omega} & , & x_o(\tau) < 0 \quad ; & x_o(\frac{\alpha\pi}{\omega}) = 0 \\ \frac{\alpha\pi}{\omega} < \tau < \frac{\pi}{\omega} & , & x_o(\tau) > 0 \quad ; & x_o(\frac{\pi}{\omega}) = A_o \end{array}\right\} \qquad (3.113)$$

Suppose $(x_o + \xi)$ is a neighboring perturbed state; evidently,

$$(x_o + \xi)'' + \{1 + g \operatorname{sgn}[(x_o + \xi)(x_o' + \xi')]\}(x_o + \xi) = P \sin(\omega\tau + \varphi_o) \qquad (3.114a)$$

with initial values,

$$x_o(0) + \xi(0) = -A_o + \xi_o \quad ; \quad x_o'(0) + \xi'(0) = \xi_o' \qquad (3.114b)$$

Subtracting (3.112a) from (3.114a),

$$\xi'' + \xi + g\{|x_o+\xi|\ \mathrm{sgn}(x_o'+\xi') - |x_o|\ \mathrm{sgn}\, x_o'\} = 0 \tag{3.115}$$

The asymptotic stability of the origin in the ξ-ξ' plane, a singular point for the differential equation in (3.115), shall guarantee the stability of the periodic solution $x_o(\tau)$, just as the results in Section III.3.

Solution during the first half-cycle

Consistent with the preliminaries in Equations (3.112), (3.113) and (3.114), during the interval $0 \leq \tau \leq \pi/\omega$, it is necessary to examine the non-linear terms in (3.115) only for three possible situations;

a) when $x_o' \to 0$ and $|\xi'|$ determines the function $\mathrm{sgn}(x_o' + \xi')$

b) when $x_o \to 0$ and ξ contributes to or dominates the expression $|x_o+\xi|$

c) whenever $|x_o|$ and $|x_o'|$ are predominant, or in other words $|x_o| >> |\xi|$, $|x_o'| >> |\xi'|$

Case a: $x_o' \to 0$

As ξ and ξ' are perturbational quantities, consistent with the properties of $x_o(\tau)$ as in (3.113) and assuming $\xi_o' > 0$, it can be concluded that during the entire half-cycle $0 \leq \tau \leq \pi/\omega$, except during a small interval at the end, about $\tau = \pi/\omega$, $\mathrm{sgn}(x_o'+\xi') = \mathrm{sgn}\, x_o' = 1$. Furthermore, without loss of generality, one might consider the case,

$$x_o'(\tau) + \xi'(\tau) > 0 \quad , \quad 0 < \tau < \left(\frac{\pi}{\omega} + \delta\right) \tag{3.116a}$$

with

$$x_o'\left(\frac{\pi}{\omega} + \delta\right) + \xi'\left(\frac{\pi}{\omega} + \delta\right) = 0 \tag{3.116b}$$

and

$$x_o'(\tau)+\xi'(\tau)<0 \quad , \quad \left(\frac{\pi}{\omega}+\delta\right)<\tau<\frac{2\pi}{\omega} \tag{3.116c}$$

while $x_o'(\tau)>0,\ 0<\tau<\frac{\pi}{\omega}$ and $x_o'(\tau)<0,\ \frac{\pi}{\omega}<\tau<\frac{2\pi}{\omega}$.

During the interval $\frac{\pi}{\omega}<\tau<(\frac{\pi}{\omega}+\delta)$, $x_o(\tau)\simeq A_o$, its positive extremal value, so that $|x_o+\xi|\simeq A_o$ also, during the same period. Then from (3.115),

$$\xi''+\xi+gA_o\{\operatorname{sgn}(x_o'+\xi')-\operatorname{sgn}x_o'\}=0 \quad , \quad \frac{\pi}{\omega}<\tau<\left(\frac{\pi}{\omega}+\delta\right) \tag{3.117}$$

Expanding (3.116b) in a series about $\tau=\frac{\pi}{\omega}$,

$$x_o'\left(\frac{\pi}{\omega}\right)+x_o''\left(\frac{\pi^+}{\omega}\right)\cdot\delta+\xi'\left(\frac{\pi}{\omega}\right)+\xi''\left(\frac{\pi^+}{\omega}\right)\cdot\delta+O(\delta^2)=0 \tag{3.118}$$

where a positive superscript denotes that the quantity under consideration shall be evaluated as its limiting value from the right.

From Equations (3.112), (3.113) and (3.117),

$$x_o''\left(\frac{\pi^+}{\omega}\right)=-(1-g)A_o+P\sin(\pi+\varphi_o)=-\{(1-g)A_o+P\sin\varphi_o\} \tag{3.119}$$

$$\xi''\left(\frac{\pi^+}{\omega}\right)=-\xi\left(\frac{\pi}{\omega}\right)-2gA_o\simeq-2gA_o \tag{3.120}$$

Since $x'\left(\frac{\pi}{\omega}\right)=0$, from (3.118), (3.119) and (3.120),

$$\delta=\frac{-\xi'\left(\frac{\pi}{\omega}\right)}{x_o''\left(\frac{\pi^+}{\omega}\right)+\xi''\left(\frac{\pi^+}{\omega}\right)}=\frac{+\xi'\left(\frac{\pi}{\omega}\right)}{(1+g)A_o+P\sin\varphi_o} \tag{3.121}$$

omitting quantities of $O(\delta^2)$. Hence

$$\begin{aligned}\xi'\left(\frac{\pi}{\omega}+\delta\right)&=\xi'\left(\frac{\pi}{\omega}\right)+\xi''\left(\frac{\pi^+}{\omega}\right)\cdot\delta+O(\delta^2)\\&=\xi'\left(\frac{\pi}{\omega}\right)-\frac{2gA_o\cdot\xi'\left(\frac{\pi}{\omega}\right)}{(1+g)A_o+P\sin\varphi_o}\end{aligned} \tag{3.122}$$

omitting quadratic and higher terms in perturbational quantities. Rewriting (3.122),

$$\left[\xi'\left(\frac{\pi}{\omega}+\delta\right)-\xi'\left(\frac{\pi}{\omega}\right)\right]=\frac{-2gA_o}{(1+g)A_o+P\sin\varphi_o}\,\xi'\left(\frac{\pi}{\omega}\right) \tag{3.123}$$

Similarly,

$$\begin{aligned}\xi\left(\frac{\pi}{\omega}+\delta\right) &= \xi\left(\frac{\pi}{\omega}\right)+\xi'\left(\frac{\pi}{\omega}\right)\cdot\delta+O(\delta^2)\\ &= \xi\left(\frac{\pi}{\omega}\right),\quad \text{correct to terms of } O\left(|\xi'|^2\right)\end{aligned} \tag{3.124}$$

Hence the effect of the terms $\operatorname{sgn}(x_o'+\xi')$ and $\operatorname{sgn}(x_o')$ during an interval when $|\xi'|$ dominates or is comparable to $|x_o'|$, may be interpreted as a jump in ξ', such that

$$\left.\begin{aligned}\xi'\left(\frac{\pi^+}{\omega}\right)-\xi'\left(\frac{\pi^-}{\omega}\right) &= \frac{-2gA_o}{(1+g)A_o+P\sin\varphi_o}\,\xi'\left(\frac{\pi^-}{\omega}\right)\\ \text{or}\quad \xi'\left(\frac{\pi^+}{\omega}\right) &= \frac{(1-g)A_o+P\sin\varphi_o}{(1+g)A_o+P\sin\varphi_o}\,\xi'\left(\frac{\pi^-}{\omega}\right)=\beta\,\xi'\left(\frac{\pi^-}{\omega}\right)\end{aligned}\right\} \tag{3.125}$$

where a negative superscript indicates the limiting value from the left, of the quantity under consideration.

The same result as in (3.125) is obtained if one were to consider that $(x_o'+\xi')$ attains zero at $\tau=(\frac{\pi}{\omega}-\delta)$ and is negative subsequently as against the case in (3.116).

Case b: $x_o \to 0$

Having considered the effect of the non-linear terms in (3.115), when the velocities $(x_o'+\xi')$ and x_o' are opposite in sign, it remains to examine the case when they are both positive, which occurs during almost the entire half-cycle mentioned above. During this period, the perturbation equation (3.115) becomes

$$\xi'' + \xi + g(|x_o + \xi| - |x_o|) = 0 \quad , \quad 0 < \tau < \frac{\pi}{\omega}^{-} \tag{3.126}$$

Recalling $x_o(\frac{\alpha\pi}{\omega}) = 0$, it is obvious that during a small interval about $\tau = \frac{\alpha\pi}{\omega}$, $|\xi|$ would dominate or at least be comparable to $|x_o|$.

Suppose

$$(x_o + \xi) < 0 \quad , \quad 0 < \tau < \left(\frac{\alpha\pi}{\omega} - \epsilon\right) \tag{3.127a}$$

$$(x_o + \xi) > 0 \quad , \quad \tau > \left(\frac{\alpha\pi}{\omega} - \epsilon\right) \tag{3.127b}$$

with

$$x_o\left(\frac{\alpha\pi}{\omega} - \epsilon\right) + \xi\left(\frac{\alpha\pi}{\omega} - \epsilon\right) = 0 \tag{3.127c}$$

From (3.126) and (3.127),

$$\xi'' + (1+g)\xi = 2g\,|x_o(\tau)| \quad , \quad \left(\frac{\alpha\pi}{\omega} - \epsilon\right) < \tau < \frac{\alpha\pi}{\omega} \tag{3.128}$$

Expanding (3.127c) in a series,

$$x_o\left(\frac{\alpha\pi}{\omega}\right) - x_o'\left(\frac{\alpha\pi}{\omega}\right)\cdot\epsilon + \xi\left(\frac{\alpha\pi}{\omega} - \epsilon\right) + O(\epsilon^2) = 0 \tag{3.129}$$

Moreover, since $x_o(\tau)$ and $\{x_o(\tau) + \xi(\tau)\}$ are both negative for $0 < \tau < (\frac{\alpha\pi}{\omega} - \epsilon)$, from (3.126),

$$\xi'' + (1-g)\xi = 0 \quad , \quad 0 < \tau < \left(\frac{\alpha\pi}{\omega} - \epsilon\right) \tag{3.130}$$

Therefore,

$$\begin{aligned}\xi\left(\frac{\alpha\pi}{\omega} - \epsilon\right) &= \xi_o \cos\left[\sqrt{1-g}\left(\frac{\alpha\pi}{\omega} - \epsilon\right)\right] + \frac{1}{\sqrt{1-g}}\,\xi_o' \sin\left[\sqrt{1-g}\left(\frac{\alpha\pi}{\omega} - \epsilon\right)\right] \\ &= \xi_o \cos\sqrt{1-g}\,\frac{\alpha\pi}{\omega} + \frac{1}{\sqrt{1-g}}\,\xi_o' \sin\sqrt{1+g}\,\frac{\alpha\pi}{\omega}\end{aligned} \tag{3.131}$$

omitting quadratic terms in ξ_o, ξ_o', and ϵ.

From (3. 129) and (3. 131),

$$\epsilon = \frac{\left(\xi_o \cos\sqrt{1-g}\,\frac{\alpha\pi}{\omega} + \frac{1}{\sqrt{1-g}}\,\xi_o' \sin\sqrt{1-g}\,\frac{\alpha\pi}{\omega}\right)}{x_o'\left(\frac{\alpha\pi}{\omega}\right)} \tag{3.132}$$

omitting quadratic terms in ξ_o, ξ_o' and ϵ. Then,

$$\begin{aligned} \xi\left(\frac{\alpha\pi}{\omega}\right) &= \xi\left(\frac{\alpha\pi}{\omega}-\epsilon\right) + \xi'\left[\left(\frac{\alpha\pi}{\omega}-\epsilon\right)^+\right]\cdot\epsilon + \cdots\cdots \\ &= \xi\left(\frac{\alpha\pi}{\omega}-\epsilon\right) + O\left(\|\underline{\xi}\|^2\right) \end{aligned} \tag{3.133}$$

Similarly,

$$\xi'\left(\frac{\alpha\pi}{\omega}\right) = \xi'\left(\frac{\alpha\pi}{\omega}-\epsilon\right) + \xi''\left[\left(\frac{\alpha\pi}{\omega}-\epsilon\right)^+\right]\cdot\epsilon + \cdots\cdots$$

But from (3. 128),

$$\begin{aligned} \xi''\left[\left(\frac{\alpha\pi}{\omega}-\epsilon\right)^+\right] &= -(1+g)\xi + 2g\left|x_o\left(\frac{\alpha\pi}{\omega}-\epsilon\right)\right| \\ &= O(\xi) \end{aligned}$$

since $|x_o(\tau)| = O(\xi)$ also, in the interval $\left(\frac{\alpha\pi}{\omega}-\epsilon\right) < \tau < \frac{\alpha\pi}{\omega}$. Therefore,

$$\xi'\left(\frac{\alpha\pi}{\omega}\right) = \xi'\left(\frac{\alpha\pi}{\omega}-\epsilon\right) + O\left(\|\underline{\xi}\|^2\right) \tag{3.135}$$

<u>Case c:</u> $|x_o| >> |\xi|$, $|x_o'| >> |\xi'|$

The results in (3. 133) and (3. 135) imply that it does not matter if one neglects to make use of (3. 127) and (3. 128) and thus uniformly consider the perturbation Equation (3. 126) to be

$$\xi'' + (1 + g\,\mathrm{sgn}\,x_o)\xi = 0 \quad , \quad 0 < \tau < \frac{\pi^-}{\omega} \tag{3.136}$$

over the half-cycle with a jump in the velocity given by (3. 125) at the end when $\tau = \frac{\pi}{\omega}$. Equation (3. 136) is identical to (3. 91) and it is possible to write down the result,

$$\begin{bmatrix} \xi\left(\frac{\pi^-}{\omega}\right) \\ \xi'\left(\frac{\pi^-}{\omega}\right) \end{bmatrix} = \begin{bmatrix} u_2\left(\frac{\pi}{\omega}\right) & u_1\left(\frac{\pi}{\omega}\right) \\ u_2'\left(\frac{\pi}{\omega}\right) & u_1'\left(\frac{\pi}{\omega}\right) \end{bmatrix} \begin{bmatrix} \xi(0) \\ \xi'(0) \end{bmatrix} \tag{3.137}$$

where $u_2(\frac{\pi}{\omega})$, $u_2'(\frac{\pi}{\omega})$, $u_1(\frac{\pi}{\omega})$ and $u_1'(\frac{\pi}{\omega})$ are obtained from (3.110) with $\tau_o=0$. Taking into account the "jump condition" in (3.125),

$$\begin{bmatrix} \xi\left(\frac{\pi^+}{\omega}\right) \\ \xi'\left(\frac{\pi^+}{\omega}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \xi\left(\frac{\pi^-}{\omega}\right) \\ \xi'\left(\frac{\pi^-}{\omega}\right) \end{bmatrix} \tag{3.138a}$$

or

$$\begin{bmatrix} \xi\left(\frac{\pi^+}{\omega}\right) \\ \xi'\left(\frac{\pi^+}{\omega}\right) \end{bmatrix} = \begin{bmatrix} u_2\left(\frac{\pi}{\omega}\right) & u_1\left(\frac{\pi^-}{\omega}\right) \\ \beta u_2'\left(\frac{\pi}{\omega}\right) & \beta u_1'\left(\frac{\pi}{\omega}\right) \end{bmatrix} \underline{\xi}(0)$$

$$= T\,\underline{\xi}(0) \tag{3.138b}$$

<u>Stability condition</u>

During the next half-cycle,

$$x_o' < 0 \quad , \quad (x_o' + \xi') < 0 \quad , \quad \frac{\pi}{\omega} < \tau < \frac{2\pi^-}{\omega} \tag{3.139}$$

except for a small interval about $\tau = \frac{2\pi}{\omega}$, when the "velocity jump"

$$\xi'\left(\frac{2\pi^+}{\omega}\right) = \beta\,\xi'\left(\frac{2\pi^-}{\omega}\right)$$

occurs with the same value of β as in (3.125). Furthermore, following the same arguments as in the first half-cycle, in the interval $\frac{\pi}{\omega} < \tau < \frac{2\pi^-}{\omega}$, the perturbation equation remains the same as in (3.136) so that,

$$\underline{\xi}\left(\frac{2\pi^+}{\omega}\right) = T\,\underline{\xi}\left(\frac{\pi^+}{\omega}\right) = T^2\,\underline{\xi}(0)$$

Similarly

$$\underline{\xi}\left(\frac{n\pi^+}{\omega}\right) = T^n \underline{\xi}(0) \tag{3.140}$$

A necessary and sufficient condition that ξ, ξ' tend to zero as $n \to \infty$, is that the eigenvalues of the matrix T be less than unity in absolute value. From (3.125), (3.138b) and (3.110), the characteristic equation for the eigenvalues is given by

$$\lambda^2 + \frac{2\lambda}{(1+g)A_o + P\sin\varphi_o}\left[(A_o + P\sin\varphi_o)\cos\beta\cos\eta - \frac{P\sin\varphi_o + (1-g^2)A_o}{\sqrt{1-g^2}}\sin\beta\sin\eta\right] + \frac{(1-g)A_o + P\sin\varphi_o}{(1+g)A_o + P\sin\varphi_o} = 0 \tag{3.141}$$

which is identical to Equation (3.111) derived in Section III. 3. Further details concerning the stability of the periodic solution of a single degree of freedom dynamic system have already been discussed.

Incidentally, it may be observed that the product of the eigenvalues λ_1 and λ_2 of the matrix T satisfies the relation

$$\lambda_1\lambda_2 = \beta = \frac{(1-g)A_o + P\sin\varphi_o}{(1+g)A_o + P\sin\varphi_o} \tag{3.142}$$

For, the product of the eigenvalues is given by the determinant of the matrix T; and from (3.138b) it follows that the determinant value is β since the Wronskian of the fundamental solutions u_1 and u_2 in (3.138b) is unity.

Hence a necessary condition for stability is that $|\beta|$ be less than 1; while this is a sufficient condition also if the roots of (3.141) are complex, it is not so if the roots are real and hence the eigenvalues must be explicitly evaluated from (3. 141) to determine the stability of the periodic solution.

It may be remarked that the analysis of the perturbation equation (3. 115) as in this section, shall be closely followed to investigate the stability of the periodic response in multi-degree of freedom systems. However, there is an essential difference in the nature of the "jump condition" in Equation (3. 125) and the "jump matrix" in (3. 138b). In the single degree of freedom system, these quantities are independent of the fact whether the perturbed velocity $(x_o' + \xi')$ attained zero earlier to or later than the unperturbed velocity; this is not true of systems with several degrees of freedom as will be seen in Section IV. 2.

III. 5 Periodic Solution by Approximate Analysis

Since the existence of a periodic solution with the same period as the sinusoidal excitation is guaranteed from previous discussion, it is naturally interesting to examine whether the periodic solution or an approximation can be derived by techniques other than having to solve the transcendental equation in (3.75). In this context, the method of slowly varying parameters is resorted to, following its successful application to solve piecewise-linear systems in earlier instances (see for example, Caughey[23]).

Let

$$x = C\cos\omega\tau + S\sin\omega\tau \tag{3.143}$$

where C and S are slowly varying functions of τ, represent the periodic solution of the problem

$$x'' + (1 + g\,\mathrm{sgn}\,xx')x = P\sin(\omega\tau + \psi_o) \tag{3.144a}$$

$$x(0) = a \quad ; \quad x'(0) = b \quad ; \quad 0 < g << 1 \tag{3.144b}$$

From (3.143)

$$x' = -\omega C\sin\omega\tau + \omega S\cos\omega\tau + C'\cos\omega\tau + S'\sin\omega\tau \tag{3.145}$$

The auxiliary equation shall be taken to be

$$C'\cos\omega\tau + S'\sin\omega\tau = 0 \tag{3.146}$$

Then

$$x'' = -\omega^2[C\cos\omega\tau + S\sin\omega\tau] - \omega C'\sin\omega\tau + \omega S'\cos\omega\tau \tag{3.147}$$

Substituting into (3.144a),

$$(1-\omega^2)[C\cos\omega\tau + S\sin\omega\tau] - \omega C'\sin\omega\tau + \omega S'\cos\omega\tau$$
$$+ g\,|C\cos\omega\tau + S\sin\omega\tau|\,\mathrm{sgn}(-C\sin\omega\tau + S\cos\omega\tau) = P\sin(\omega\tau+\psi_o) \tag{3.148}$$

Multiplying both sides of (3.146) by $\omega \sin \omega\tau$, both sides of (3.148) by $\cos \omega\tau$ and adding,

$$(1-\omega^2)[C\cos^2\omega\tau + S\sin\omega\tau\cos\omega\tau] + \omega S' + g\sqrt{C^2+S^2}\,|\sin(\omega\tau+\gamma)|\,\mathrm{sgn}\Big(\cos(\omega\tau+\gamma)\Big)\cos\omega\tau = P\cos\omega\tau\sin(\omega\tau+\psi_o) \tag{3.149}$$

where

$$\tan\gamma = \frac{C}{S}$$

Averaging over a cycle,

$$(1-\omega^2)C + 2\omega S' + \frac{g\sqrt{C^2+S^2}}{\pi}\int_0^{2\pi}|\sin(\theta+\gamma)|\,\mathrm{sgn}\Big(\cos(\theta+\gamma)\Big)\cos\theta\,d\theta = P\sin\psi_o \tag{3.150}$$

Similarly,

$$(1-\omega^2)S - 2\omega C' + \frac{g\sqrt{C^2+S^2}}{\pi}\int_0^{2\pi}|\sin(\theta+\gamma)|\,\mathrm{sgn}\Big(\cos(\theta+\gamma)\Big)\sin\theta\,d\theta = P\cos\psi_o \tag{3.151}$$

To evaluate integrals in (3.150) and (3.151), let

$$I_1 = \int_0^{2\pi}|\sin\lambda|\,\mathrm{sgn}(\cos\lambda)\sin\lambda\,d\lambda = 0 \tag{3.152a}$$

$$I_2 = \int_0^{2\pi}|\sin\lambda|\,\mathrm{sgn}(\cos\lambda)\cos\lambda\,d\lambda = 4\int_0^{\pi/2}\sin\lambda\cos\lambda\,d\lambda = 2 \tag{3.152b}$$

Then

$$\int_0^{2\pi}|\sin(\theta+\gamma)|\,\mathrm{sgn}\{\cos(\theta+\gamma)\}\cos\theta\,d\theta = I_2\cos\gamma + I_1\sin\gamma = 2\cos\gamma \tag{3.153a}$$

$$\int_0^{2\pi} |\sin(\theta+\gamma)| \operatorname{sgn}\{\cos(\theta+\gamma)\} \sin\theta \, d\theta = I_1 \cos\gamma - I_2 \sin\gamma = -2\sin\gamma \quad (3.153b)$$

Therefore,

$$(1-\omega^2)C + 2\omega S' + \frac{2g}{\pi}\sqrt{C^2+S^2}\cos\gamma = P\sin\psi_o \quad (3.154a)$$

$$(1-\omega^2)S - 2\omega C' - \frac{2g}{\pi}\sqrt{C^2+S^2}\sin\gamma = P\cos\psi_o \quad (3.154b)$$

Since $\tan\gamma = C/S$, (3.154) reduces to

$$\left.\begin{aligned}(1-\omega^2)C + 2\omega S' + \frac{2g}{\pi}S &= P\sin\psi_o \\ (1-\omega^2)S - 2\omega C' - \frac{2g}{\pi}C &= P\cos\psi_o\end{aligned}\right\} \quad (3.155)$$

From (3.143) and (3.144),

$$x(0) = C(0) = a \quad ; \quad x'(0) = S(0)\omega = b \quad (3.156)$$

Let

$$Z = C + iS \quad (3.157)$$

Then substituting (3.157) into (3.155) and (3.156),

$$\left.\begin{aligned}(1-\omega^2)Z - 2i\omega Z' - \frac{2ig}{\pi}Z &= iPe^{-i\psi_o} \\ Z(0) &= a + \frac{ib}{\omega}\end{aligned}\right\} \quad (3.158)$$

Solving the first order differential equation in (3.158),

$$Z(\tau) = \left\{a + \frac{ib}{\omega} + \frac{Pe^{-i(\psi_o+\varphi)}}{\sqrt{(1-\omega^2)^2 + \left(\frac{2g}{\pi}\right)^2}}\right\} e^{-\left\{\frac{2g}{\pi} + i(1-\omega^2)\right\}\frac{\tau}{2\omega}} - \frac{Pe^{-i(\psi_o+\varphi)}}{\sqrt{(1-\omega^2)^2 + \left(\frac{2g}{\pi}\right)^2}} \quad (3.159)$$

where $\tan\varphi = (1-\omega^2)/(2g/\pi)$.

The complete solution is given by

$$x(\tau) = [\mathrm{Re}\, Z(\tau)] \cos \omega\tau + [\mathrm{Im}\, Z(\tau)] \sin \omega\tau \tag{3.160}$$

In particular, as $\tau \to \infty$,

$$\underset{\tau \to \infty}{\mathfrak{L}t}\ Z(\tau) = - \frac{P e^{-i(\psi_o + \varphi)}}{\sqrt{(1-\omega^2)^2 + \left(\frac{2g}{\pi}\right)^2}} \tag{3.161}$$

Moreover, selecting $\psi_o = \pi$,

$$\underset{\tau \to \infty}{\mathfrak{L}t}\ x(\tau) = A \cos(\omega\tau + \varphi) = \frac{P \cos(\omega\tau + \varphi)}{\sqrt{(1-\omega^2)^2 + \left(\frac{2g}{\pi}\right)^2}}$$

$$= \frac{P}{\sqrt{(1-\omega^2)^2 + \left(\frac{2g}{\pi}\right)^2}} \sin(\omega\tau + \psi_o - \varphi_s) \tag{3.162}$$

where

$$\varphi_s = \frac{\pi}{2} - \varphi \quad , \quad \tan \varphi = \frac{(1-\omega^2)}{(2g/\pi)} \tag{3.163}$$

Thus the response to an excitation

$$f(\tau) = P \sin(\omega\tau + \psi_o) \tag{3.164}$$

is given by

$$x(\tau) = A \sin(\omega\tau + \psi_o - \varphi_s) \tag{3.165}$$

where

$$\frac{A}{P} = \frac{1}{\sqrt{(1-\omega^2)^2 + \left(\frac{2g}{\pi}\right)^2}} \tag{3.166}$$

and the "phase-shift φ_s" is given in Equation (3.163). The phase-shift so obtained is truly the phase difference between the fundamental harmonic response, obtained by approximation, and the sinusoidal input.

The above solution is of course <u>stable</u> since it was obtained as the limit of the transient solution as $\tau \to \infty$.

The method of slowly varying parameters is a one-term or fundamental harmonic approximation only in the general asymptotic scheme (see Minorsky[24], pg. 360, and Bogoliubov and Mitropolsky[25], pg. 134). Hence it is considered relevant to make a comparative study of the above results in Equations (3. 163) and (3. 166) with only the fundamental harmonic in the exact solution derived in Section IV. 2 .

Let

$$x_h(\tau) = a_1 \cos \omega\tau + b_1 \sin \omega\tau \tag{3. 167}$$

be the fundamental harmonic contained in the exact solution $x(\tau)$ represented in Equations (3. 64) and (3. 69). Evidently,

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} x(\tau) \cos \omega\tau \, d(\omega\tau) \tag{3. 168a}$$

and

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} x(\tau) \sin \omega\tau \, d(\omega\tau) \tag{3. 168b}$$

From the symmetry property $x(\tau) = -x(\tau + \pi/\omega)$,

$$a_1 = \frac{2}{\pi}\left\{\int_0^{\alpha\pi} x(\theta) \cos \theta \, d\theta + \cos \alpha\pi \int_0^{(1-\alpha)\pi} x(\theta') \cos \theta' d\theta' - \sin \alpha\pi \int_0^{(1-\alpha)\pi} x(\theta') \sin \theta' d\theta'\right\} \tag{3. 169a}$$

$$b_1 = \frac{2}{\pi}\left\{\int_0^{\alpha\pi} x(\theta) \sin \theta \, d\theta + \cos \alpha\pi \int_0^{(1-\alpha)\pi} x(\theta') \sin \theta' d\theta' + \sin \alpha\pi \int_0^{(1-\alpha)\pi} x(\theta') \cos \theta' d\theta'\right\} \tag{3. 169b}$$

$x(\theta)$ is obtained from (3. 64) on substituting $\omega\tau = \theta$ and similarly $x(\theta')$ from (3. 69) on substituting $\omega\tau = (\theta' + \alpha\pi)$. Equation (3. 167) may be rewritten as follows:

$$\left.\begin{aligned} x_h(\tau) = A_h \sin(\omega\tau + \varphi^*) = A_h \sin(\omega\tau + \varphi_o - \varphi_h) \\ A_h = \sqrt{a_1^2 + b_1^2} \quad , \quad a_1 = A_h \sin\varphi^* \quad , \quad b_1 = A_h \cos\varphi^* \quad ; \quad \varphi_h = \varphi_o - \varphi^* \end{aligned}\right\} \quad (3.170)$$

where φ_o is the "phase" obtained from solving (3.76). Thus φ_h is the actual phase shift between the fundamental harmonic in the response and the input.

The amplitudes from (3.166) and (3.170) are plotted in Figure 4 and the phase angles φ_s and φ_h in Figure 5. There is excellent agreement in general although the maximum error, which occurs for $g = 0.2$, is as high as 10% in amplitude and 8% in phase.

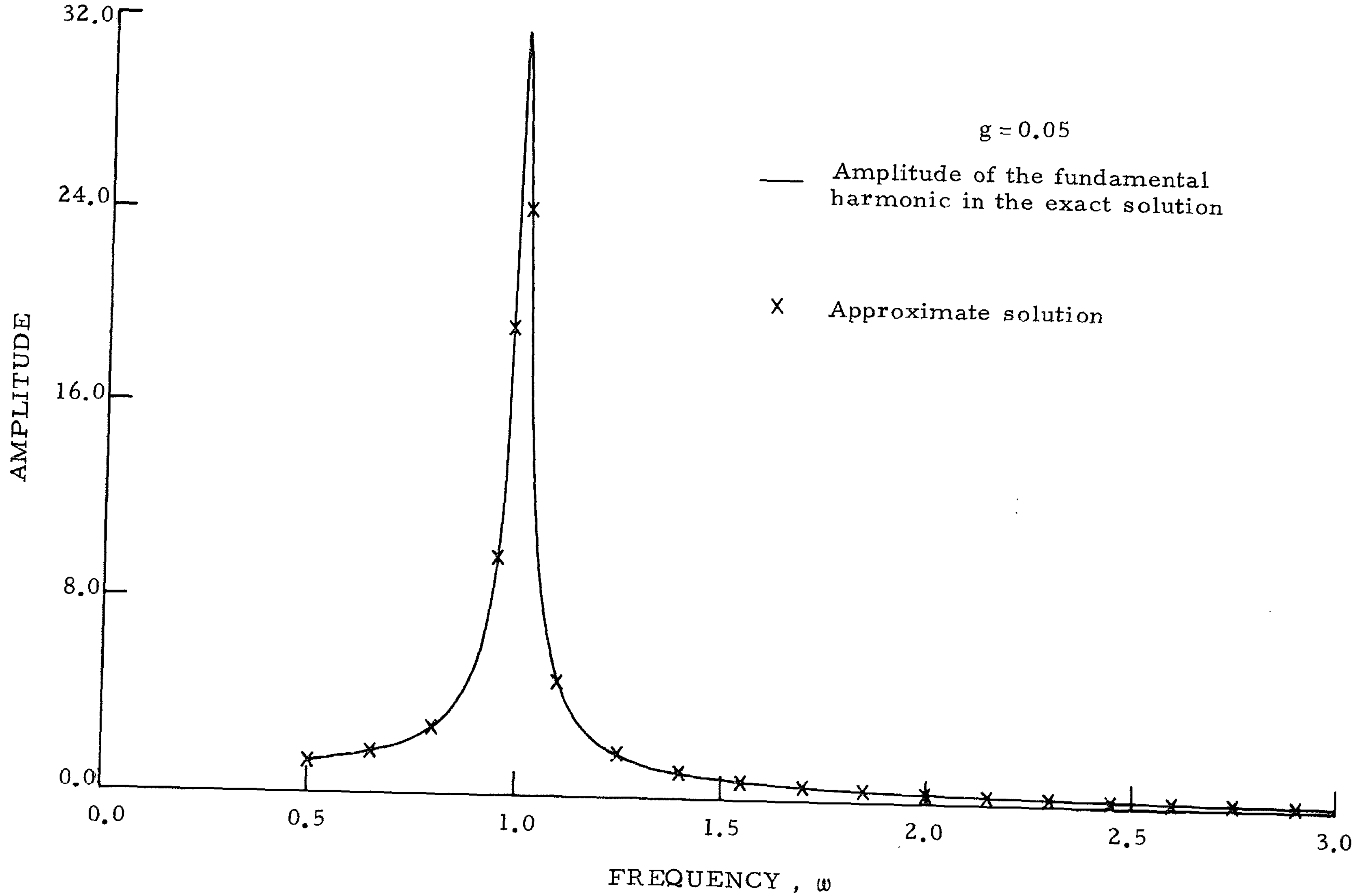


Figure 4a. Comparison of the Exact and Approximate Solutions

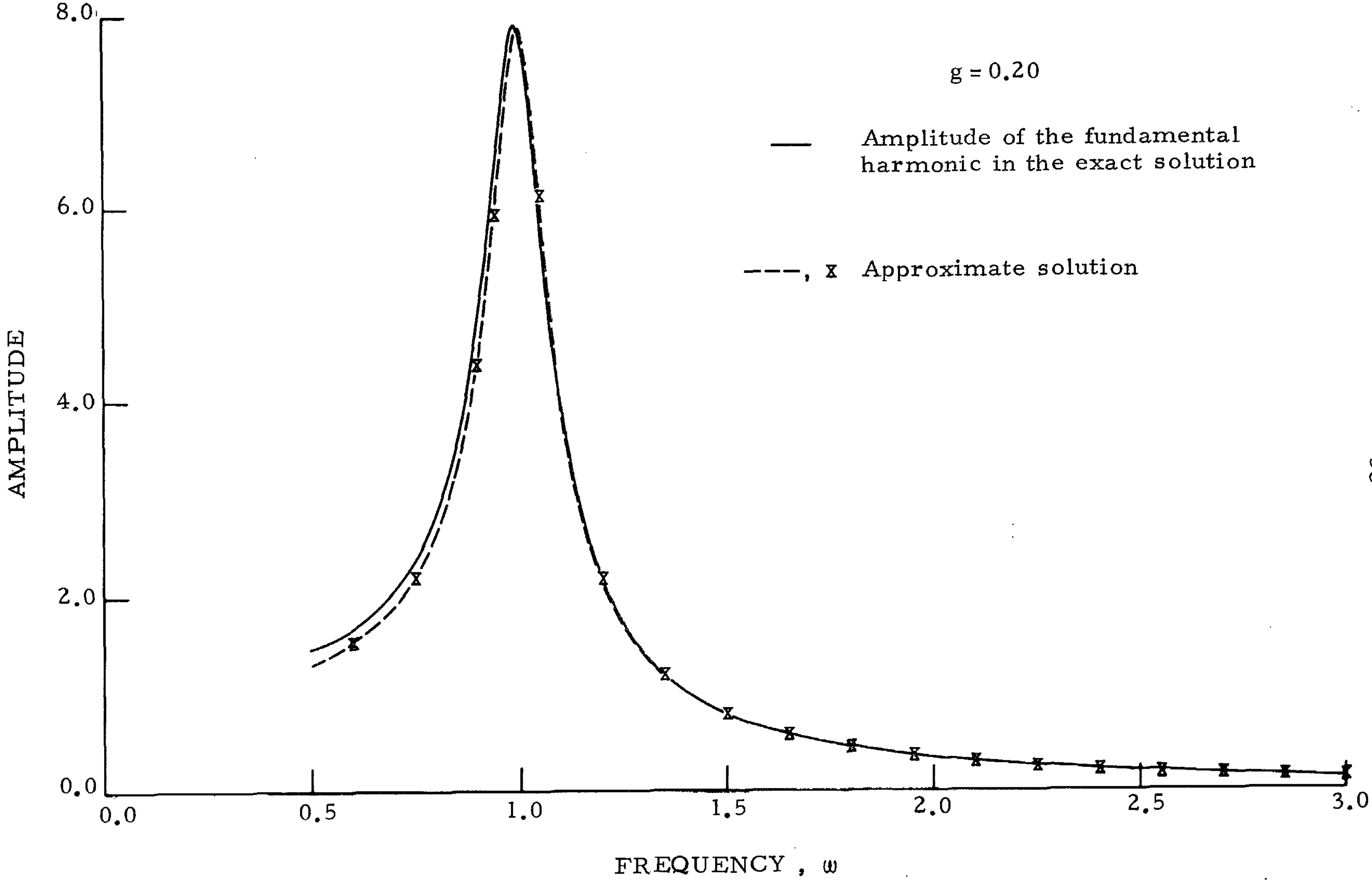


Figure 4b. Comparison of the Exact and Approximate Solutions

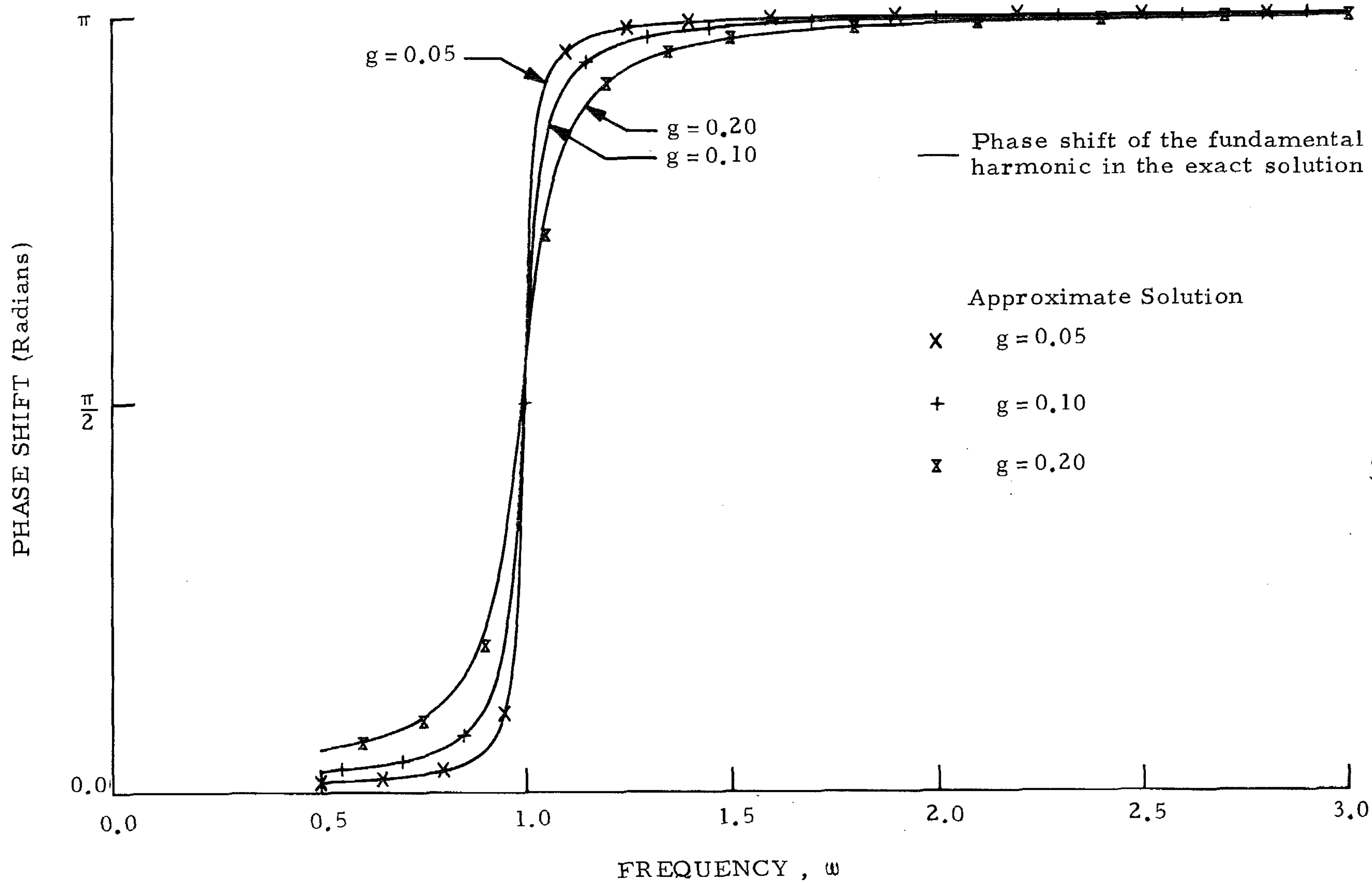


Figure 5. Comparison of the Exact and Approximate Solutions

CHAPTER IV

FORCED OSCILLATIONS IN MULTI-DEGREE OF FREEDOM SYSTEMS

IV. 1 Exact Solutions for a Restricted Class of Excitation

A dynamic system with N-degrees of freedom is schematically shown in Figure 6. R_1, $R_2 \cdots R_{N+1}$ are all "Reid springs", which during cyclic motion give rise to energy dissipation because of their piecewise-linear, non-linear characteristics. In this chapter, it is desired to study some aspects of the forced oscillation in such systems. In so far as exact solutions are sought, the analysis presented here is confined to only certain cases with very specific "modal excitation", to be described subsequently. The restriction is essentially due to the complexity of the non-linear terms, which makes it impossible to uncouple the differential equations of motion by coordinate transformations, for arbitrary modes of excitation, as is done in linear systems.

Equations of motion

The equations of motion are given by

$$m_k\ddot{x}_k + s_k\{1 + g \operatorname{sgn}[(x_k - x_{k-1})(\dot{x}_k - \dot{x}_{k-1})]\}(x_k - x_{k-1})$$
$$+ s_{k+1}\{1 + g \operatorname{sgn}[(x_k - x_{k+1})(\dot{x}_k - \dot{x}_{k+1})]\}(x_k - x_{k+1}) = F_k\varphi(t) \quad k=1,2,3\ldots N \tag{4.1}$$

where m_k is the k-th mass, s_k is the spring constant of the k-th spring, and x_k is the displacement of the k-th mass with x_o and x_{N+1} being stipulated identically zero.

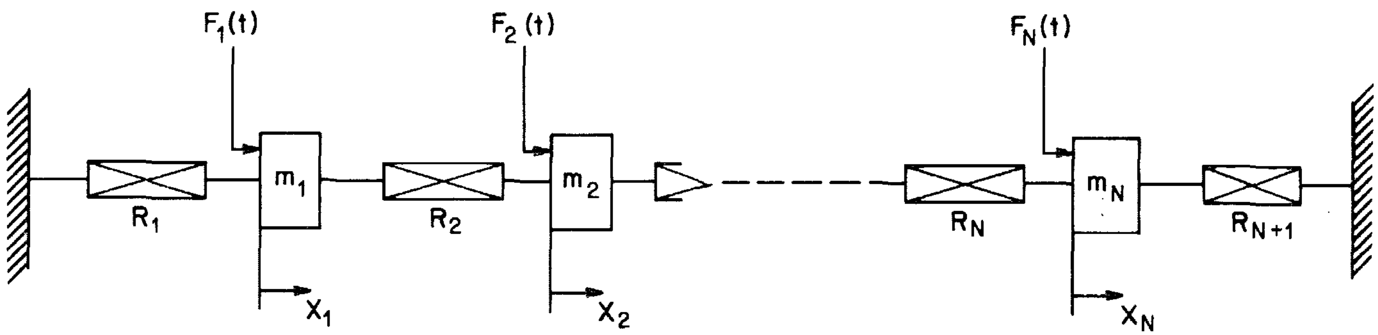


Figure 6. A Schematic Representation of a Multi-Degree of Freedom System

The force vector is specifically chosen to be of the form $\underline{F}\varphi(t)$ where $\underline{F}$ is a constant vector independent of time, $\varphi(t)$ is a scalar function in time to be subsequently considered trigonometric.

Equation (4.1) shall be rewritten in the matrix-vector notation as follows

$$M\underline{\ddot{x}} + S\underline{x} + gS^{*}(\underline{x})\underline{x} = \underline{F}\varphi(t) \tag{4.2}$$

where M is the diagonal mass-matrix with

$$M_{ij} = m_i \delta_{ij} \quad , \quad \delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \tag{4.3}$$

The tri-diagonal matrices S and $S^{*}(\underline{x})$ consist of the linear and non-linear terms respectively of the general spring matrix associated with the restoring force represented in (4.1).

The only three non-zero elements of the k-th row of the matrix S are

$$S_{k,k-1} = -s_k \quad ; \quad S_{k,k} = s_k + s_{k+1} \quad ; \quad S_{k,k+1} = -s_{k+1} \tag{4.4}$$

Similarly,

$$\left.\begin{aligned} S^{*}_{k,k-1}(\underline{x}) &= -s_k \operatorname{sgn}\{(x_k - x_{k-1})(\dot{x}_k - \dot{x}_{k-1})\} \\ S^{*}_{k,k}(\underline{x}) &= s_k \operatorname{sgn}\{(x_k - x_{k-1})(\dot{x}_k - \dot{x}_{k-1})\} \\ &\quad + s_{k+1} \operatorname{sgn}\{(x_k - x_{k+1})(\dot{x}_k - \dot{x}_{k+1})\} \\ S^{*}_{k,k+1}(\underline{x}) &= -s_{k+1} \operatorname{sgn}\{(x_k - x_{k+1})(\dot{x}_k - \dot{x}_{k+1})\} \end{aligned}\right\} \tag{4.5}$$

Evidently, S, $S^{*}(\underline{x})$ are real symmetric matrices.

<u>Existence and uniqueness of solutions</u>

The restoring force in Equation (4.2) is bounded and piecewise continuous in any bounded region of the associated 2-N dimensional phase-space. In addition, let the external forces be piecewise continuous and bounded also. Then the Cauchy-Lipshitz theorem guarantees the existence of a unique solution to the initial value problem in any region of the phase-space bounded away from the discontinuity surfaces of the restoring force. In any region so described, the restoring force is continuous and for any pair of vectors $\underline{x}$ and $\underline{y}$ lying entirely in such a region

$$\|(S+S^*(\underline{x}))\underline{x}-(S+S^*(\underline{y})\underline{y})\| \leq 4(1+g)s_{max}\|\underline{x}-\underline{y}\| \tag{4.6}$$

where

$$s_{max} = \max_k s_k$$

Thus the necessary conditions of the Cauchy-Lipshitz theorem are satisfied. In fact, the solution can be continued up to the first discontinuity encountered.

The Lipshitz condition in Equation (4.6) is not satisfied only when the two vectors $\underline{x}$ and $\underline{y}$ are such that for one or more values of k, it is true that

$$\mathrm{sgn}\,(x_k - x_{k+1}) = \mathrm{sgn}\,(y_k - y_{k+1})$$

and

$$\mathrm{sgn}\,(\dot{x}_k - \dot{x}_{k+1}) = -\mathrm{sgn}\,(\dot{y}_k - \dot{y}_{k+1}) \quad , \quad k = 1,2,\cdots N$$

Hence the difficulty arises in extending the uniqueness property of the solution across such a discontinuity surface. But then, for the non-autonomous system under consideration, $\varphi(t)$ being a non-constant function, singular point solutions cannot exist. And just as in the case of a dynamic system with single degree of freedom, for large values of $\|\underline{x}\|$, the discontinuity surfaces cannot contain arcs of the solution trajectories of (4.2). This enables one to extend the solution indefinitely repeating the same arguments.

Beyond this point, the analysis of a Reid oscillator with single degree of freedom entailed a study in the phase-plane only. With just two possible values for the signum function in the non-linearity part, explicit solutions could be sought for the differential equation or first integrals evaluated as necessary, so the continuity of mapping and ultimate boundedness of solutions could be established with appropriate conditions on the external force. Furthermore, the existence of at least one periodic solution on periodic excitation was guaranteed by the application of the fixed point theorem to the phase-plane analysis. However, it is not found convenient or practicable to extend these techniques to the multi-degree of freedom problem, simply because of the 2-N dimensional character of the phase-space and the resulting very large number of permutations and combinations involved in dealing with the signum functions in Equation (4.1).

Still it is possible to obtain exact solutions in certain cases, proceeding from a knowledge of the corresponding results in linear systems.

<u>Exact solution for sinusoidal excitation</u>

The matrix-differential equation (4.2) may be rewritten in a partly canonical form. Let $M^{-1/2}$ denote the diagonal matrix with elements,

$$M_{ij}^{-1/2} = \frac{1}{\sqrt{m_i}}\ \delta_{ij} \tag{4.7}$$

where δ_{ij} is again the Kronecker-delta,

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq i \\ 1 \text{ if } i = j \end{cases}$$

and $\sqrt{m_i}$ indicates the positive root. Premultiplying both sides of (4.2) by $M^{-1/2}$ and introducing a new variable

$$\underline{x} = M^{-1/2}\underline{y} \tag{4.8}$$

$$M^{-1/2}MM^{-1/2}\ddot{\underline{y}} + M^{-1/2}SM^{-1/2}\underline{y} + g\,M^{-1/2}S^*(\underline{y})M^{-1/2}\underline{y} = M^{-1/2}\underline{F}\varphi(t) \tag{4.9a}$$

or

$$I\ddot{\underline{y}} + S_1\underline{y} + gS_1^*(\underline{y})\underline{y} = M^{-1/2}\underline{F}\varphi(t) \tag{4.9b}$$

Evidently S_1 and $S_1^*(\underline{y})$ are real symmetric.

Throughout the present work, it is assumed that the eigen values of S_1 are well separated and it has a full complement of eigen vectors. S_1 being real symmetric, its eigen values are real and the eigen vectors mutually orthogonal. For the type of spring-mass system under consideration, it is not a severe restriction to assume widely separated eigen values, if the various spring-constants are well-balanced.

Suppose that g=0; Equation (4.9b) becomes a purely linear system; in addition, let

$$M^{-1/2}\underline{F} = \omega_j^2\underline{E}^{(j)} \tag{4.10}$$

and

$$\varphi(t) = \sin(\omega t + \theta_j) \tag{4.11}$$

where $\underline{E}^{(j)}$ is the eigen vector of the matrix S_1, corresponding to its j-th eigen value $\lambda_j = \omega_j^2$; θ_j is an arbitrary constant. The proportionality factor ω_j^2 in (4. 10) is introduced to help non-dimensionalizing the equations consistently. If $\omega^2 \neq \lambda_j$, then a harmonic solution to the problem,

$$I\ddot{y} + S_1\underline{y} = \omega_j^2 \underline{E}^{(j)} \sin(\omega t + \theta_j) \tag{4. 12}$$

has exactly the same mode as $\underline{E}^{(j)}$; in fact, the particular solution referred to is just,

$$\underline{y} = \frac{1}{\left(1-\omega^2/\omega_j^2\right)} \underline{E}^{(j)} \sin(\omega t + \theta_j) \tag{4. 13}$$

In other words, for suitable initial conditions, an external force $\omega_j^2 \underline{E}^{(j)} \sin(\omega t + \theta_j)$ excites only the "corresponding pure normal mode under free vibrations". If linear viscous damping were also present, the steady-state oscillations strictly correspond to this normal mode.

The "Reid oscillators" being dissipative, the harmonic response in damped linear systems is suggestive to seek a solution of the equation

$$I\ddot{y} + S_1\underline{y} + gS_1^*(\underline{y})\underline{y} = \omega_j^2 \underline{E}^{(j)} \varphi(t) \tag{4. 14}$$

in the form,

$$\underline{y} = \underline{E}^{(j)} \xi(t) \tag{4. 15}$$

On substituting (4. 15) into (4. 14), it is seen that the vector differential equation is simply equivalent to the scalar equation,

$$\ddot{\xi} + \omega_j^2 \{1 + g \operatorname{sgn}(\xi\dot{\xi})\} \xi = \omega_j^2 \varphi(t) \tag{4. 16}$$

the k-th row of (4. 14) being obtained by multiplying both sides of (4. 16) by the constant $E_k^{(j)}$.

The uncoupling is accomplished, as the form of solution given in (4.15) implies,

$$\left(x_k - x_{k-1}\right) = \left(\frac{E_k^{(j)}}{\sqrt{m_k}} - \frac{E_{k-1}^{(j)}}{\sqrt{m_{k-1}}}\right)\xi(t) \tag{4.17}$$

so that

$$\operatorname{sgn}\{(x_k - x_{k-1})(\dot{x}_k - \dot{x}_{k-1})\} = \operatorname{sgn}(\xi\dot{\xi}) \tag{4.18}$$

Hence

$$\begin{aligned} S_1^*(\underline{y})\underline{y} &= M^{-1/2}S^*(\underline{x})\underline{x} \\ &= M^{-1/2}(\operatorname{sgn}\xi\dot{\xi})SM^{-1/2}\underline{y} \\ &= \operatorname{sgn}(\xi\dot{\xi})S_1\underline{E}^{(j)}\xi(t) \end{aligned} \tag{4.19}$$

and (4.14) becomes,

$$I\underline{E}^{(j)}\ddot{\xi}(t) + S_1\underline{E}^{(j)}\xi(t) + g\operatorname{sgn}(\xi\dot{\xi})S_1\underline{E}^{(j)}\xi(t) = \omega_j^2\underline{E}^{(j)}\varphi(t) \tag{4.20}$$

Recalling that $\underline{E}^{(j)}$ is the j-th eigen vector of S_1 such that $S_1\underline{E}^{(j)} = \omega_j^2\underline{E}^{(j)}$, (4.20) yields on equating the corresponding components on both sides,

$$\ddot{\xi} + \omega_j^2\{1 + g\operatorname{sgn}(\xi\dot{\xi})\}\xi = \omega_j^2\varphi(t) \tag{4.21}$$

The properties of the solutions of (4.21) have already been studied in Chapter III and the exact periodic solution has been derived in Section III.2, for sinusoidal excitation $\varphi(t) = \sin(\omega t + \varphi_o)$.

Thus, given $\xi(t)$ to be the exact solution of the Equation (4.21), it is seen that

$$\underline{x} = M^{-1/2}\underline{E}^{(j)}\xi(t) \tag{4.22}$$

satisfies the differential equation

$$M\underline{\ddot{x}} + S\underline{x} + gS^{*}(\underline{x})\underline{x} = \omega_j^2 M^{1/2} \underline{E}^{(j)} \varphi(t) \tag{4.23}$$

with initial values,

$$\underline{x}(0) = M^{-1/2} \underline{E}^{(j)} \xi(0) \quad ; \quad \underline{\dot{x}}(0) = M^{-1/2} \underline{E}^{(j)} \dot{\xi}(0) \tag{4.24}$$

$M^{-1/2}$ denotes the inverse of the matrix $M^{1/2}$; in particular

$$M_{ij}^{1/2} = \sqrt{m_i}\, \delta_{ij} \tag{4.25}$$

$\underline{x}(t)$ is periodic, if $\varphi(t)$ is periodic also. N such exact solutions can be obtained so long as the external periodic (trigonometric) force excites one and only one of the N normal modes, as already described.

In the foregoing discussion, $\xi(t)$ in (4.21) is known to admit of at least one periodic solution with the same period as the excitation and it has been verified by direct substitution that x(t) as in (4.22) satisfies the differential equation (4.23). However, it has not been shown that (4.23) admits x(t) as a stable periodic solution; in fact, not even the ultimate boundedness of x(t) has been established. Hence it is meaningful to consider x(t) as a periodic solution of (4.23) only if it is shown to be stable against at least infinitesimal perturbations. Thus it is again necessary to focus attention on the associated stability problem.

IV.2 Stability of the Periodic Solution in Multi-Degree of Freedom Systems

In principle, the stability analysis of the periodic response in multi-degree of freedom systems follows directly from the corresponding problem in the single degree of freedom case as formulated in Section III.4. However, here the primary task is to uncouple the perturbation equations.

<u>Canonical form of the perturbation equations</u>

Let

$$\underline{x}_o(t) = M^{-1/2}\underline{E}^{(j)}\xi(t) \tag{4.26}$$

represent the periodic solution of the equation

$$M\ddot{\underline{x}} + S\underline{x} + gS^{*}(\underline{x})\underline{x} = \omega_j^2 M^{1/2}\underline{E}^{(j)} \sin(\omega t+\theta_j) \tag{4.27}$$

where $\xi(t)$, constructed as outlined in Section III.2, satisfies

$$\ddot{\xi} + \omega_j^2\{1+g \operatorname{sgn} \xi\dot{\xi}\}\xi = \omega_j^2 \sin(\omega t+\theta_j) \tag{4.28}$$

θ_j is so chosen that

$$\xi(0) = -A_j \quad ; \quad \dot{\xi}(0) = 0 \tag{4.29}$$

Consider a neighboring perturbed state

$$\underline{x}_p = \underline{x}_o + \underline{\eta} \tag{4.30}$$

Obviously,

$$M\ddot{\underline{x}}_p + S\underline{x}_p + gS^{*}(\underline{x}_p)\underline{x}_p = \omega_j^2 M^{1/2}\underline{E}^{(j)} \sin(\omega t+\theta_j) \tag{4.31}$$

Subtracting (4.27) from (4.31), taking into account (4.30),

$$M\underline{\ddot{\eta}}+S\underline{\eta}+g\left\{[S^*(\underline{x}_o+\underline{\eta})](\underline{x}_o+\underline{\eta})-S^*(\underline{x}_o)\underline{x}_o\right\}=0 \tag{4.32}$$

Let

$$t^{(j)}_{k,k-1}=\frac{E^{(j)}_k}{\sqrt{m_k}}-\frac{E^{(j)}_{k-1}}{\sqrt{m_{k-1}}} \tag{4.33}$$

It will be observed that $t^{(j)}_{k,k-1}$ may assume a value of zero depending upon the mode excited and the masses involved. The non-linear terms in the k-th row of Equation (4.32) can be written down explicitly, making use of (4.26) and (4.33). They are,

$$\begin{aligned} s_k&\left\{\left|t^{(j)}_{k,k-1}\xi+\eta_k-\eta_{k-1}\right|\operatorname{sgn}\left(t^{(j)}_{k,k-1}\dot{\xi}+\dot{\eta}_k-\dot{\eta}_{k-1}\right)-\left|t^{(j)}_{k,k-1}\xi\right|\operatorname{sgn}\left(t^{(j)}_{k,k-1}\dot{\xi}\right)\right\}\\ &+s_{k+1}\left\{\left|t^{(j)}_{k,k+1}\xi+\eta_k-\eta_{k+1}\right|\operatorname{sgn}\left(t^{(j)}_{k,k+1}\dot{\xi}+\dot{\eta}_k-\dot{\eta}_{k+1}\right)\right.\\ &\qquad\left.-\left|t^{(j)}_{k,k+1}\xi\right|\operatorname{sgn}\left(t^{(j)}_{k,k+1}\dot{\xi}\right)\right\}=H_{k,k-1}+H_{k,k+1} \end{aligned} \tag{4.34}$$

where the notations $H_{k,k-1}$ and $H_{k,k+1}$ are self-evident.

Let E denote the matrix consisting of column vectors $\underline{E}^{(m)}$, $m=1,2,\cdots N$, where $\underline{E}^{(m)}$ is the m-th eigen vector of the matrix $S_1=M^{-1/2}SM^{-1/2}$, corresponding to its eigen value $\lambda_m=\omega_m^2$. E^T shall denote the transpose of the matrix E. Let

$$\underline{\eta}=M^{-1/2}E\underline{\mu} \tag{4.35}$$

so that

$$\eta_k=\frac{1}{\sqrt{m_k}}\sum_m E^{(m)}_k\mu_m \tag{4.36}$$

Then from Equations (4.34), (4.35) and (4.36)

$$H_{k,k-1} = s_k t^{(j)}_{k,k-1} \left\{ \left| \xi + \frac{1}{t^{(j)}_{k,k-1}} \left(\sum_{m} t^{(m)}_{k,k-1} \mu_m \right) \right| \operatorname{sgn} \left[\dot{\xi} + \frac{1}{t^{(j)}_{k,k-1}} \left(\sum_{m} t^{(m)}_{k,k-1} \dot{\mu}_m \right) \right] - |\xi| \operatorname{sgn} \dot{\xi} \right\} \quad (4.37)$$

assuming $t^{(j)}_{k,k-1} \neq 0$, $k = 2, 3 \cdots (N+1)$.

Rewriting (4.37)

$$H_{k,k-1} = s_k t^{(j)}_{k,k-1} \left\{ \left| \xi + \left(\sum_{m} \gamma^{(m)}_{k,k-1} \mu_m \right) \right| \operatorname{sgn} \left[\dot{\xi} + \left(\sum_{m} \gamma^{(m)}_{k,k-1} \dot{\mu}_m \right) \right] - |\xi| \operatorname{sgn} \dot{\xi} \right\} \quad (4.38)$$

where

$$\gamma^{(m)}_{k,k-1} = \frac{t^{(m)}_{k,k-1}}{t^{(j)}_{k,k-1}}$$

This is of the form

$$H_{k,k-1} = s_k t^{(j)}_{k,k-1} \left\{ f\left(\xi + \sum_{m} \alpha_m \mu_m \right) - f(\xi) \right\} \quad (4.39)$$

where

$$f(\xi) = |\xi| \operatorname{sgn} \dot{\xi} \quad (4.40)$$

Recalling that the perturbations being considered are infinitesimal, it is necessary to retain linear terms only in the perturbation quantities. In particular, it may be observed

$$f\left(\xi + \sum_{m} \alpha_m \mu_m \right) - f(\xi) = \sum_{m} \alpha_m \left\{ f(\xi + \mu_m) - f(\xi) \right\} + O(\|\underline{\mu}\|^2) \quad (4.41)$$

so that

$$\left| \xi + \left(\sum_{m} \gamma^{(m)}_{k,k-1} \mu_m \right) \right| \operatorname{sgn} \left[\dot{\xi} + \left(\sum_{m} \gamma^{(m)}_{k,k-1} \dot{\mu}_m \right) \right] - |\xi| \operatorname{sgn} \dot{\xi}$$

$$= \sum_{m} \gamma^{(m)}_{k,k-1} \left\{ |\xi + \mu_m| \operatorname{sgn}(\dot{\xi} + \dot{\mu}_m) - |\xi| \operatorname{sgn} \dot{\xi} \right\} + O(\|\underline{\mu}\|^2) \quad (4.42)$$

In deriving (4.42), it is to be remembered that although the functions under consideration possess only generalized derivatives, correct to first order terms in the perturbation quantities μ_m, $m=1,2,\cdots N$, the expressions on either side of the equality sign are identical and hence can be interchanged in formulating the linear perturbation problem. Substituting (4.42) into (4.38),

$$H_{k,k-1} = s_k t^{(j)}_{k,k-1} \sum_m \frac{t^{(m)}_{k,k-1}}{t^{(j)}_{k,k-1}} \left\{ |\xi+\mu_m| \operatorname{sgn}(\dot{\xi}+\dot{\mu}_m) - |\xi| \operatorname{sgn}\dot{\xi} \right\}$$

$$= s_k (M^{-1/2}E\underline{\rho})_k - s_k (M^{-1/2}E\underline{\rho})_{k-1} \tag{4.43}$$

where $\underline{\rho}$ is the vector $\{\rho_m\}$,

$$\rho_m = |\xi+\mu_m| \operatorname{sgn}(\dot{\xi}+\dot{\mu}_m) - |\xi| \operatorname{sgn}\dot{\xi} \tag{4.44}$$

Finally, the non-linear terms in the k-th row of (4.32) shall be written,

$$H_{k,k-1} + H_{k,k+1} = -s_k (M^{-1/2}E\underline{\rho})_{k-1} + s_k (M^{-1/2}E\underline{\rho})_k$$

$$+ s_{k+1}(M^{-1/2}E\underline{\rho})_k - s_{k+1}(M^{-1/2}E\underline{\rho})_{k+1} \tag{4.45}$$

Substituting (4.45) and (4.35) into (4.32) and pre-multiplying by $(E^T M^{-1/2})$,

$$I\ddot{\underline{\mu}} + \Lambda\underline{\mu} + g\Lambda\underline{\rho} = 0 \tag{4.46}$$

where Λ is the diagonal matrix with elements

$$\Lambda_{ij} = \omega_i^2 \, \delta_{ij} \tag{4.47}$$

ω_i^2 is the i-th eigenvalue of $S_i = M^{-1/2} S M^{-1/2}$ and δ_{ij} is the Kronecker-delta. From (4.46) it follows,

$$\ddot{\mu}_k + \omega_k^2 \mu_k + g\omega_k^2 \left\{ |\xi+\mu_k| \operatorname{sgn}(\dot{\xi}+\dot{\mu}_k) - |\xi| \operatorname{sgn}\dot{\xi} \right\} = 0 \quad k = 1,2\cdots N \tag{4.48}$$

which is indeed in the canonical form desired.

It must however be remarked that the canonical form above has been derived strictly under the assumption that $t_{k,k-1}^{(j)} \neq 0$, $k = 2,3\cdots(N+1)$. If this condition is not satisfied during a particular modal excitation, the perturbation equations cannot be uncoupled in this way. Still it may be possible to determine the stability of the periodic solution in particular cases as is shown in the next section in Example 1, for a two-degree of freedom system excited in the "in-phase" modal oscillation.

Stability analysis

The asymptotic stability of the origin in the phase-plane for each of the N equations in (4.48) shall guarantee the stability of the periodic solution $\underline{x}_o(t)$. Then a typical case to examine is

$$\ddot{\mu}_k + \omega_k^2 \mu_k + g\omega_k^2 \left\{ |\xi+\mu_k| \operatorname{sgn}(\dot{\xi}+\dot{\mu}_k) - |\xi| \operatorname{sgn}\dot{\xi} \right\} = 0 \tag{4.49}$$

ξ satisfies

$$\ddot{\xi} + \omega_j^2 \left\{ 1+g \operatorname{sgn} \xi\dot{\xi} \right\} \xi = \omega_j^2 \sin(\omega t+\theta_j) \tag{4.50}$$

Furthermore, from previous results

$$\left.\begin{array}{c}
\xi(0) = -A_j \quad ; \quad \xi(t) < 0 \;,\quad 0 < t < \dfrac{\alpha^*\pi}{\omega} \quad ; \quad \xi\left(\dfrac{\alpha^*\pi}{\omega}\right) = 0 \quad ; \\
\xi(t) > 0 \;,\quad \dfrac{\alpha^*\pi}{\omega} < t < \dfrac{\pi}{\omega} \quad ; \quad \xi\left(\dfrac{\pi}{\omega}\right) = A_j \quad ; \\
\dot{\xi}(0) = \dot{\xi}\left(\dfrac{\pi}{\omega}\right) = 0 \quad ; \quad \dot{\xi}(t) > 0 \;,\quad 0 < t < \dfrac{\pi}{\omega} \quad ; \\
\xi\left(t + \dfrac{\pi}{\omega}\right) = -\xi(t)
\end{array}\right\} \tag{4.51}$$

where α^* is evaluated by solving the transcendental equation (3.75) corresponding to the frequency $\omega^* = \frac{\omega}{\omega_j}$, which is necessary to reduce (4.50) to the non-dimensional form in (3.61). The equations (4.49) and (4.50) are similar to (3.115) and (3.112a), the perturbation and the basic differential equation respectively for a single degree of freedom dynamic system.

Closely following the analysis in Section III.4, it is necessary to investigate in detail the propagation of the initial perturbations only during the first half-cycle, including the "velocity jump" at the end, when $t = \frac{\pi}{\omega}$. In carrying out the required steps, quadratic and higher degree terms in perturbational quantities shall be omitted; similarly linear terms in perturbational quantities shall be neglected in comparison with the maximum absolute values of displacement or velocity of the unperturbed solution. These details shall not be explicitly stated any further in this section.

The "velocity jump" at $t = \pi/\omega$

Consistent with the preliminaries in Equations (4.49), (4.50) and (4.51), it is evident that in a small interval about $t = \frac{\pi}{\omega}$, $\dot{\mu}_k$ will undergo a sharp change in its value, interpreted as a jump in this work.

Case a

Let

$$(\dot{\xi}+\mathring{\mu}_k) > 0 \quad , \quad 0 < t < \left(\frac{\pi}{\omega}+\delta\right) \tag{4.52a}$$

$$(\dot{\xi}+\dot{\mu}_k) < 0 \quad , \quad \left(\frac{\pi}{\omega}+\delta\right) < t < \frac{2\pi}{\omega} \tag{4.52b}$$

with

$$\dot{\xi}\left(\frac{\pi}{\omega}+\delta\right)+\dot{\mu}_k\left(\frac{\pi}{\omega}+\delta\right)=0 \tag{4.53}$$

Since $\dot{\xi}(t)<0$ for $\frac{\pi}{\omega}<t<\frac{2\pi}{\omega}$, consistent with (4.52a), $\dot{\mu}_k>0$ in the interval $\frac{\pi}{\omega}<t<\left(\frac{\pi}{\omega}+\delta\right)$. From (4.49),

$$\ddot{\mu}_k(t)=-2g\omega_k^2 A_j \quad , \quad \frac{\pi}{\omega}<t<\left(\frac{\pi}{\omega}+\delta\right) \tag{4.54}$$

Expanding (4.53) in a series,

$$\delta=\frac{-\dot{\mu}_k\left(\frac{\pi}{\omega}+\delta\right)}{\ddot{\xi}\left(\frac{\pi^+}{\omega}\right)}=\frac{\dot{\mu}_k\left(\frac{\pi}{\omega}+\delta\right)}{\omega_k^2\{(1-g)A_j+\sin\theta_j\}} \tag{4.55}$$

taking into account (4.50) and (4.51) also. But

$$\begin{aligned}\dot{\mu}_k\left(\frac{\pi}{\omega}+\delta\right)&=\dot{\mu}_k\left(\frac{\pi}{\omega}\right)+\ddot{\mu}_k\left(\frac{\pi^+}{\omega}\right)\delta\\&=\dot{\mu}_k\left(\frac{\pi}{\omega}\right)-2g\omega_k^2 A_j\delta\\&=\dot{\mu}_k\left(\frac{\pi}{\omega}\right)-\frac{2gA_j}{(1-g)A_j+\sin\theta_j}\frac{\omega_k^2}{\omega_j^2}\dot{\mu}_k\left(\frac{\pi}{\omega}+\delta\right)\end{aligned} \tag{4.56}$$

from (4.54) and (4.55). Interpreting this result as a jump, just as was done in the case of a single degree of freedom system,

$$\left.\begin{aligned}\dot{\mu}_k\left(\frac{\pi^+}{\omega}\right)&=\frac{1}{1+\dfrac{2gA_j}{(1-g)A_j+\sin\theta_j}\dfrac{\omega_k^2}{\omega_j^2}}\dot{\mu}_k\left(\frac{\pi^-}{\omega}\right)\\&=\beta_+\dot{\mu}_k\left(\frac{\pi^-}{\omega}\right)\end{aligned}\right\} \tag{4.57}$$

where the positive subscript for β denotes that the perturbed velocity velocity $(\dot{\xi}+\dot{\mu}_k)$ attains zero at $t>\frac{\pi}{\omega}$, or rather at some instant later than the velocity $\dot{\xi}$ corresponding to the unperturbed periodic solution. It is observed that $0 \leq \beta_+ \leq 1$, in the entire range of the ratio ω_k^2/ω_j^2 as this ratio decreases from ∞ to 0.

<u>Case b</u>

Contrary to the hypothesis in (4.52) in Case a, let

$$(\dot{\xi}+\dot{\mu}_k) > 0 \quad , \quad 0 < t < \left(\frac{\pi}{\omega} - \delta\right) \tag{4.58a}$$

$$(\dot{\xi}+\dot{\mu}_k) < 0 \quad , \quad \left(\frac{\pi}{\omega} - \delta\right) < t < \frac{2\pi}{\omega} \tag{4.58b}$$

with

$$\dot{\xi}\left(\frac{\pi}{\omega} - \delta\right) + \dot{\mu}_k\left(\frac{\pi}{\omega} - \delta\right) = 0 \tag{4.59}$$

since $\dot{\xi} > 0$ in the interval $0 < t < \frac{\pi}{\omega}$, from (4.58b) and (4.59), <u>necessarily</u>

$$\dot{\mu}_k(t) < 0 \quad , \quad \left(\frac{\pi}{\omega} - \delta\right) \leq t \leq \frac{\pi}{\omega} \tag{4.60}$$

Expanding (4.59) in a series and taking into consideration (4.50) and (4.51),

$$\delta = \frac{\dot{\mu}_k\left(\frac{\pi}{\omega} - \delta\right)}{\ddot{\xi}\left(\frac{\pi}{\omega}^-\right)} = \frac{-\dot{\mu}_k\left(\frac{\pi}{\omega} - \delta\right)}{\omega_j^2\{(1+g)A_j + \sin\theta_j\}} \tag{4.61}$$

Moreover,

$$\dot{\mu}_k\left(\frac{\pi}{\omega} - \delta\right) = \dot{\mu}_k\left(\frac{\pi}{\omega}\right) - \ddot{\mu}_k\left(\frac{\pi}{\omega}^-\right)\delta$$

$$= \dot{\mu}_k\left(\frac{\pi}{\omega}\right) + \frac{2gA_j}{(1+g)A_j + \sin\theta_j}\,\frac{\omega_k^2}{\omega_j^2}\,\dot{\mu}_k\left(\frac{\pi}{\omega} - \delta\right)$$

from Equations (4.49) and (4.61). Thus finally

$$\left.\begin{aligned} \dot{\mu}_k\left(\frac{\pi^+}{\omega}\right) &= \left\{1 - \frac{2gA_j}{(1+g)A_j + \sin\theta_j}\,\frac{\omega_k^2}{\omega_j^2}\right\}\dot{\mu}_k\left(\frac{\pi^-}{\omega}\right) \\ &= \beta_-\,\dot{\mu}_k\left(\frac{\pi^-}{\omega}\right) \end{aligned}\right\} \tag{4.62}$$

The negative subscript for β shall denote that the perturbed velocity $(\dot{\xi}+\dot{\mu}_k) < 0$ for $t > \left(\frac{\pi}{\omega} - \delta\right)$ while the unperturbed velocity $\dot{\xi} > 0$ in $0 < t < \frac{\pi}{\omega}$. As the ratio ω_k^2/ω_j^2 increases from 0, β_- decreases from 1, till eventually $\beta_- \leq 0$ for

$$\frac{\omega_k^2}{\omega_j^2} > \frac{(1+g)A_j + \sin\theta_j}{2gA_j}$$

However, <u>negative values for β are irrelevant</u>, as this would imply that $\dot{\mu}_k > 0$ in $\left(\frac{\pi}{\omega} - \delta\right) < t < \frac{\pi}{\omega}$, which in turn indicates a contradiction to the hypothesis as stated in (4.58) and (4.60); in other words, for sufficiently large values of ω_k^2/ω_j^2, if $\beta_- \leq 0$, the velocity cross-over as in Case a shall only be considered.

It is of interest to note that $\beta_+ = \beta_-$ if $\omega_k^2 = \omega_j^2$, which is the result earlier obtained in Section III. 4.

<u>Perturbation solution in the interval $0 < t < \pi/\omega$</u>

Already having considered the case when $\dot{\xi}$ and $(\dot{\xi}+\dot{\mu}_k)$ are opposite in sign at the end of the interval $0 < t < \frac{\pi}{\omega}$, when $t = \frac{\pi}{\omega}$, it remains to solve the perturbation equation (4.49) when

$$\operatorname{sgn}\dot{\xi} = \operatorname{sgn}(\dot{\xi}+\dot{\mu}_k) = +1$$

which is the case in almost the entire half-cycle, prior to the instant when the jump in velocity occurs. During this interval, from (4.49),

$$\ddot{\mu}_k + \omega_k^2 \mu_k + g\omega_k^2 \{|\xi + \mu_k| - |\xi|\} = 0 \tag{4.63}$$

Since $\xi\left(\frac{\alpha^* \pi}{\omega}\right) = 0$, in a small interval about $t = \frac{\alpha^* \pi}{\omega}$, $|\mu_k| = O(|\xi|)$; however, following the arguments in Section III. 4 in deriving Equation (3. 136), it is not essential to treat this case in detail separately. Thus, the perturbation equation (4. 63) shall be uniformly considered as

$$\ddot{\mu}_k + \omega_k^2 \{1 + g \operatorname{sgn} \xi\} \mu_k = 0 \quad , \quad 0 < t < \frac{\pi^-}{\omega} \tag{4.64}$$

with $\xi(t) < 0$, $0 < t < \frac{\alpha^* \pi}{\omega}$ and $\xi(t) > 0$, $\frac{\alpha^* \pi}{\omega} < t < \frac{\pi}{\omega}$ as in (4. 51). Solving (4. 64),

$$\begin{bmatrix} \mu_k\left(\frac{\pi^-}{\omega}\right) \\ \dot{\mu}_k\left(\frac{\pi^-}{\omega}\right) \end{bmatrix} = T \begin{bmatrix} \mu_k(0) \\ \dot{\mu}_k(0) \end{bmatrix} \tag{4.65}$$

where

$$\left.\begin{aligned}
T_{11} &= \cos \eta_2 \cos \eta_1 - \sqrt{\frac{1-g}{1+g}} \sin \eta_2 \sin \eta_1 \\
T_{12} &= \frac{1}{\omega_k \sqrt{1-g}} \cos \eta_2 \sin \eta_1 + \frac{1}{\omega_k \sqrt{1+g}} \sin \eta_2 \cos \eta_1 \\
T_{21} &= -\omega_k \left\{ \sqrt{1+g} \sin \eta_2 \cos \eta_1 + \sqrt{1-g} \cos \eta_2 \sin \eta_1 \right\} \\
T_{22} &= -\sqrt{\frac{1+g}{1-g}} \sin \eta_2 \sin \eta_1 + \cos \eta_2 \cos \eta_1 \\
\eta_1 &= \omega_k \sqrt{1-g} \, \frac{\alpha^* \pi}{\omega} \quad , \quad \eta_2 = \omega_k \sqrt{1+g} \, \frac{(1-\alpha^*)\pi}{\omega}
\end{aligned}\right\} \tag{4.66}$$

<u>Stability conditions</u>

Taking into consideration the "jump conditions" in either one of the Equations (4. 57) or (4. 62) as applicable,

$$\begin{bmatrix} \mu_k\left(\frac{\pi^+}{\omega}\right) \\ \dot{\mu}_k\left(\frac{\pi^+}{\omega}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix} T\underline{\mu}_k(0) = T_1\underline{\mu}_k(0) \tag{4.67}$$

where

$$\underline{\mu}_k = \begin{bmatrix} \mu_k \\ \dot{\mu}_k \end{bmatrix} \quad \text{and} \quad \beta = \beta_+ \text{ or } \beta_-$$

As before, a sufficient condition for the stability of the periodic solution $\underline{x}_o(t)$ is that the eigen values of the matrix T_1 be less than 1 in absolute value.

From (4.66) and (4.67), the characteristic polynomial for the eigne values is given by

$$\lambda^2 - \lambda\left\{\cos\eta_2 \cos\eta_1 - \sqrt{\frac{1-g}{1+g}} \sin\eta_1 \sin\eta_2 + \beta\left(\cos\eta_2\cos\eta_1 - \sqrt{\frac{1+g}{1-g}}\sin\eta_2\sin\eta_1\right)\right\} + \beta = 0 \tag{4.68}$$

where β assumes either one of the values

$$\beta = \beta_+ = \frac{1}{1 + \dfrac{2gA_j}{(1-g)A_j + \sin\theta_j}\dfrac{\omega_k^2}{\omega_j^2}} \tag{4.69a}$$

or

$$\beta = \beta_- = \left\{1 - \frac{2gA_j}{(1+g)A_j + \sin\theta_j}\frac{\omega_k^2}{\omega_j^2}\right\} \tag{4.69b}$$

where β_- in (4.69b) is relevant if and only if $0 < \beta_- < 1$. Hence if $\omega_k^2 \neq \omega_j^2$, it is necessary to evaluate the roots of the two quadratic equations resulting from substituting (4.69a) and (4.69b) successively into

(4.68). Each of the four eigen values of T_1 so obtained (or only two if β_- as in (4.69b) is irrelevant and β_+ only is applicable) must be less than unity in modulus.

Furthermore, such a stability analysis for the asymptotic stability of the origin in the phase-plane must be carried out for each of the N equations in (4.48) to ensure the stability of the periodic solution $\underline{x}_o(t)$, one excited mode of a dynamic system with N degrees of freedom.

IV. 3 Analysis of a Two-Degree of Freedom System

In order to illustrate the principles and procedures discussed in the preceding two sections, two examples shall be worked out concerning the periodic solutions in a dynamic system with two degrees of freedom.

Consider the forced oscillations in a two-degree of freedom spring-mass system with equal masses and identical "Reid-springs". Let the external sinusoidal forces acting on the two masses have the same amplitude and frequency.

<u>Example 1</u>

In addition, let the external forces be <u>exactly in-phase.</u> The equations of motion are

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ g\begin{bmatrix} \operatorname{sgn} x_1 \dot{x}_1 + \operatorname{sgn} x_d \dot{x}_d & -\operatorname{sgn} x_d \dot{x}_d \\ -\operatorname{sgn} x_d \dot{x}_d & \operatorname{sgn} x_2 \dot{x}_2 + \operatorname{sgn} x_d \dot{x}_d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(\omega t + \theta_1) \qquad (4.70)$$

where $x_d = (x_1 - x_2)$. Or

$$\underline{I}\,\underline{\ddot{x}} + \underline{S}\,\underline{x} + gS^*(\underline{x})\underline{x} = \underline{f}_1 \sin(\omega t + \theta_1) \qquad (4.71)$$

implying obvious correspondence with the terms in (4.70). It may be observed that $\underline{f}_1$ is an eigen vector of the matrix S; in fact, it is the eigen vector corresponding to the eigen value $\lambda_1 = \omega_1^2 = 1$ of the matrix S.

Following the results in Section IV. 1,

$$\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi(t) \tag{4.72}$$

is a periodic solution of (4. 71), where $\xi(t)$ satisfies

$$\ddot{\xi} + \{1+g \operatorname{sgn} \xi\dot{\xi}\}\xi = \sin(\omega t+\theta_1) \tag{4.73}$$

θ_1 is so chosen that $\xi(0) = -A_1$; $\dot{\xi}(0) = 0$ and $\xi(t)$ is the exact periodic solution as constructed in Section III. 2 .

The stability analysis of this mode of excitation <u>does not strictly follow</u> the derivation given above in Section IV. 2; as per the notation in (4. 33),

$$t^{(1)}_{1,2} = 0 \tag{4.74}$$

and hence this is the exceptional case. In the perturbed state, let

$$x_1 = \xi + \eta_1 \quad , \quad x_2 = \xi + \eta_2 \tag{4.75}$$

The perturbation equations corresponding to (4. 32) are

$$\ddot{\eta}_1 + 2\eta_1 - \eta_2 + g\left\{\left[|\xi+\eta_1| \operatorname{sgn}(\dot{\xi}+\dot{\eta}_1) - |\xi| \operatorname{sgn}\dot{\xi}\right] + |\eta_1-\eta_2| \operatorname{sgn}(\dot{\eta}_1-\dot{\eta}_2)\right\} = 0 \tag{4.76a}$$

$$\ddot{\eta}_2 + 2\eta_2 - \eta_1 + g\left\{\left[|\xi+\eta_2| \operatorname{sgn}(\dot{\xi}+\dot{\eta}_2) - |\xi| \operatorname{sgn}\dot{\xi}\right] + |\eta_2-\eta_1| \operatorname{sgn}(\dot{\eta}_2-\dot{\eta}_1)\right\} = 0 \tag{4.76b}$$

The terms $|\eta_1-\eta_2| \operatorname{sgn}(\dot{\eta}_1-\dot{\eta}_2)$, $|\eta_2-\eta_1| \operatorname{sgn}(\dot{\eta}_2-\dot{\eta}_1)$ in (4. 76) appearing without the corresponding unperturbed terms indicate the exceptional situation in this excitation. However, in a two-degree of freedom system, these terms involve one of the principal coordinates of the system and hence it is still possible to uncouple the Equations (4. 76a) and (4. 76b).

Let

$$\underline{\eta} = E\underline{\mu} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underline{\mu} \tag{4.77}$$

where E diagonalizes the matrix S. Substituting (4.77) into (4.76a) and approximating the non-linear terms as in (4.41),

$$\begin{aligned} &|\xi+\eta_1|\,\mathrm{sgn}\,(\dot{\xi}+\dot{\eta}_1) - |\xi|\,\mathrm{sgn}\,\dot{\xi} + |\eta_1-\eta_2|\,\mathrm{sgn}\,(\dot{\eta}_1-\dot{\eta}_2) \\ &\quad = \left\{|\xi+\mu_1+\mu_2|\,\mathrm{sgn}\,(\dot{\xi}+\dot{\mu}_1+\dot{\mu}_2) - |\xi|\,\mathrm{sgn}\,\dot{\xi}\right\} + 2|\mu_2|\,\mathrm{sgn}\,\dot{\mu}_2 \\ &\quad \simeq \left\{|\xi+\mu_1|\,\mathrm{sgn}\,(\dot{\xi}+\dot{\mu}_1) - |\xi|\,\mathrm{sgn}\,\dot{\xi}\right. \\ &\qquad + \left\{|\xi+\mu_2|\,\mathrm{sgn}\,(\dot{\xi}+\dot{\mu}_2) - |\xi|\,\mathrm{sgn}\,\dot{\xi}\right\} + 2|\mu_2|\,\mathrm{sgn}\,\dot{\mu}_2 \\ &\quad = \rho_1 + \rho_2 + 2|\mu_2|\,\mathrm{sgn}\,\dot{\mu}_2 \end{aligned} \tag{4.78}$$

where

$$\rho_i = |\xi+\mu_i|\,\mathrm{sgn}\,(\dot{\xi}+\dot{\mu}_i) - |\xi|\,\mathrm{sgn}\,\dot{\xi} \quad , \quad i = 1, 2 \tag{4.79}$$

Similarly,

$$|\eta_2-\eta_1|\,\mathrm{sgn}\,(\dot{\eta}_2-\dot{\eta}_1) + |\xi+\eta_2|\,\mathrm{sgn}\,(\dot{\xi}+\dot{\eta}_2) - |\xi|\,\mathrm{sgn}\,\dot{\xi} \simeq \rho_1 - \rho_2 - 2|\mu_2|\,\mathrm{sgn}\,\dot{\mu}_2 \tag{4.80}$$

Substituting (4.79) and (4.80) into (4.76),

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\underline{\eta}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \underline{\eta} + g \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underline{\rho} + 2g \begin{bmatrix} 1 \\ -1 \end{bmatrix} |\mu_2|\,\mathrm{sgn}\,\dot{\mu}_2 = 0 \tag{4.81}$$

From (4.81),

$$\ddot{\mu}_1 + \mu_1 + g\rho_1 = 0 \tag{4.82a}$$

$$\ddot{\mu}_2 + 3\mu_2 + g\rho_2 + 2g|\mu_2|\,\mathrm{sgn}\,\dot{\mu}_2 = 0 \tag{4.82b}$$

Or finally,

$$\ddot{\mu}_1 + \mu_1 + g\left\{|\xi+\mu_1|\,\mathrm{sgn}\,(\dot{\xi}+\dot{\mu}_1) - |\xi|\,\mathrm{sgn}\,\dot{\xi}\right\} = 0 \tag{4.83}$$

$$\ddot{\mu}_2 + 3\mu_2 + 2g|\mu_2| \operatorname{sgn} \dot{\mu}_2 + g\left\{|\xi+\mu_2| \operatorname{sgn}(\dot{\xi}+\dot{\mu}_2) - |\xi| \operatorname{sgn} \dot{\xi}\right\} = 0 \qquad (4.84)$$

Equation (4.83) has been studied in detail in Sections III.3 and III.4 and the asymptotic stability of the origin assured for the range of parameters considered therein.

It may be observed that (4.84) is the perturbation equation corresponding to $\omega_2^2 = 3$, for the excited mode corresponding to $\omega_1^2 = 1$.

In the analysis of Equation (4.84), once again the non-linear terms $\{|\xi+\mu_2| \operatorname{sgn}(\dot{\xi}+\dot{\mu}_2) - |\xi| \operatorname{sgn} \dot{\xi}\}$ shall be interpreted to contribute to a "jump in velocity" at $t = \pi/\omega$ such that

$$\dot{\mu}_2\left(\frac{\pi}{\omega}^+\right) = \left\{1 - \frac{2gA_1}{(1+g)A_1 + \sin\theta_1}\right\} \dot{\mu}_2\left(\frac{\pi}{\omega}^-\right) \qquad (4.85)$$

for both cases as in (a) and (b), corresponding to β_+ and β_- in Equations (4.57) and (4.62).

For subsequent considerations, the perturbation equation (4.84) shall be uniformly considered to be

$$\ddot{\mu}_2 + 3\mu_2 + g\mu_2 \operatorname{sgn} \xi + 2g|\mu_2| \operatorname{sgn} \dot{\mu}_2 = 0 \quad 0 < t < \frac{\pi}{\omega} \qquad (4.86)$$

Since $0 < \beta_+ = \beta_- < 1$, for the values of $g = 0.05$, 0.1 and 0.2, for $\omega > 0.5$ the trajectories of (4.84) spiral inwards to the origin, if the trajectories of (4.86) do so. Hence a sufficient condition for the asymptotic stability of the origin in the phase-plane for (4.84) is to show that the trajectories of (4.86) spiral inwards to the origin.

Let

$$\tau = \sqrt{3}\, t \quad , \quad \mu_2' = \frac{d\mu_2}{d\tau} \quad , \quad \mu_2'' = \frac{d^2\mu}{d\tau^2} \qquad (4.87)$$

Then (4.86) becomes

$$\mu_2'' + \mu_2 + g_2\mu_2 \operatorname{sgn} \mu_2\mu_2' + g_1\mu_2 \operatorname{sgn} \xi = 0 \tag{4.88}$$

where

$$g_2 = \frac{2}{3}g \quad , \quad g_1 = \frac{1}{3}g \tag{4.89}$$

Let

$$\mu_2\mu_2' < 0 \;,\; \tau_0 < \tau < \tau_1 \quad ; \quad \mu_2'(\tau_o) = 0 \quad ; \quad \mu_2(\tau_1) = 0 \quad ; \quad \mu_2'(\tau) > 0 \tag{4.90}$$

Suppose that

$$V_1 = \frac{1}{2}\mu_2'^2 + \frac{1}{2}(1-g_2+g_1)\mu_2^2 \tag{4.91}$$

$V_1 \geq 0$, where equality holds if and only if $\mu_2 = \mu_2' = 0$. Along the trajectories of (4.88),

$$V_1' = \mu_2'\mu_2'' + (1-g_2+g_1)\mu_2\mu_2'$$

$$= -g_1\mu_2\mu_2'\{\operatorname{sgn} \xi - 1\}$$

$$= \begin{cases} 0 & \text{if} \quad \operatorname{sgn} \xi = 1 \\ -2g_1|\mu_2\mu_2'| & \text{if} \quad \operatorname{sgn} \xi = -1 \end{cases} \tag{4.92}$$

It may be recalled that sgn ξ can be 0 only for an instant and not during any interval. Therefore, V_1 decreases or at most remains constant during the interval $\tau_o < \tau < \tau_1$.

Suppose that

$$\begin{aligned} &\mu_2\mu_2' > 0 \;,\; \tau_1 < \tau < \tau_2 \quad ; \quad \mu_2(\tau_1) = 0 \\ &\mu_2'(\tau_2) = 0 \quad \text{and} \quad \mu_2(\tau) > 0 \;,\; \tau_1 < \tau < \tau_2 \end{aligned} \tag{4.93}$$

Let

$$V_2 = \frac{1}{2}\mu_2'^2 + \frac{1}{2}(1+g_2-g_1)\mu_2^2 \quad , \quad V_2 \geq 0$$

Along the trajectories of (4. 88),

$$
\begin{aligned}
V_2' &= -g_1\mu_2\mu_2'\{\operatorname{sgn}\xi+1\} \\
&= \begin{cases} -2g_1\,|\mu_2\mu_2'| & \text{if } \operatorname{sgn}\xi = 1 \\ 0 & \text{if } \operatorname{sgn}\xi = -1 \end{cases}
\end{aligned}
\tag{4.94}
$$

From (4. 92) and (4. 94), it is observed that the "appropriate norm" V_1 or V_2 in each quadrant, depending upon whether $\mu_2\mu_2' \lesseqgtr 0$, remains constant or decreases; but even if both remain constant,

$$
\mu_2^2(\tau_2) = \frac{1-g_2+g_1}{1+g_2-g_1}\mu_2^2(\tau_o) = \frac{1-\frac{1}{3}g}{1+\frac{1}{3}g}\mu_2^2(\tau_o)
$$

or

$$
\begin{gathered}
\mu_2^2(\tau_2) < \mu_2^2(\tau_o) \quad , \quad \text{even if} \\
V_1' = V_2' = 0
\end{gathered}
\tag{4.95}
$$

Also

$$
\mu_2'^2(\tau_1) = (1-g_2+g_1)\mu_2^2(\tau_o) = K\,\mu_2^2(\tau_o) \tag{4.96}
$$

where K is a constant. Furthermore these results are valid in the lower half of the phase-plane also, for $\mu_2' < 0$. Suppose

$$
\begin{gathered}
\mu_2\mu_2' < 0\ , \ \ \tau_2 < \tau < \tau_3 \quad ; \quad \mu_2\mu_2' > 0\ , \ \ \tau_3 < \tau < \tau_4 \quad ; \\
\mu_2(\tau_3) = 0 \quad ; \quad \mu_2'(\tau_4) = 0
\end{gathered}
$$

Then repeating the same arguments

$$
\mu_2^2(\tau_4) < \frac{\left(1-\frac{1}{3}g\right)^2}{\left(1+\frac{1}{3}g\right)^2}\mu_2^2(\tau_o) \tag{4.97}
$$

$$\mu_2'^2(\tau_3) < K \frac{\left(1-\frac{1}{3}g\right)}{\left(1+\frac{1}{3}g\right)} \mu_2^2(\tau_o)$$

or

$$\mu_2'^2(\tau_3) < \mu_2'^2(\tau_2) \qquad (4.98)$$

So from (4.97) and (4.98), it is seen that the trajectories of (4.88) necessarily spiral inwards to the origin; this is essentially due to the stabilizing influence of the terms $2g\mu_2 \operatorname{sgn}(\mu_2\dot{\mu}_2)$ in (4.88); if these terms were absent, (4.88) would have been just the well-known Hill-Meissner equation (see Den Hartog[(26)], page 387) which admits of unstable solutions.

Thus, given $|\beta| < 1$, the perturbation μ_2 is necessarily asymptotically stable at the origin and since the same result has been derived in Section III.3 concerning $\mu_1(t)$, the periodic solution $x_1 = \xi(t)$, $x_2 = \xi(t)$ is stable.

Example 2

Consider the same spring-mass system as in Example 1, but with the external forces exactly opposite in phase. This example will be treated following directly, the formulation of the stability problem in Section IV.2.

The equations of motion are

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + g\begin{bmatrix} \operatorname{sgn} x_1\dot{x}_1 + \operatorname{sgn} x_d\dot{x}_d & -\operatorname{sgn} x_d\dot{x}_d \\ -\operatorname{sgn} x_d\dot{x}_d & \operatorname{sgn} x_2\dot{x}_2 + \operatorname{sgn} x_d\dot{x}_d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin(\omega t + \theta_2) \qquad (4.99)$$

Or

$$\underline{I}\underline{\ddot{x}} + S\underline{x} + gS^{*}(\underline{x})\underline{x} = 3\underline{f}_2 \sin(\omega t + \theta_2) \tag{4.100}$$

It may be observed that $\underline{f}_2$ is the eigen vector of the matrix S, corresponding to its eigen value $\lambda_2 = \omega_2^2 = 3$. Just as before,

$$\underline{x} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \xi(t) \tag{4.101}$$

is a periodic solution of (4.100), if $\xi(t)$ satisfies

$$\ddot{\xi} + 3\{1 + g \operatorname{sgn} \xi\dot{\xi}\}\xi = 3 \sin(\omega t + \theta_2) \tag{4.102}$$

θ_2 is so chosen that $\xi(0) = -A_2$; $\dot{\xi}(0) = 0$; in particular, for $\omega > 0.5\sqrt{3}$, the values of A_2 and θ_2 are obtained from the construction given in III.2. The second mode being excited in this example,

$$t_{1,2}^{(2)} = \frac{E_1^{(2)}}{\sqrt{m_1}} - \frac{E_2^{(2)}}{\sqrt{m_2}} = 2$$

which assures that the general formulation in Section IV.2 is valid in this case.

The perturbation equations corresponding to (4.32) are

$$\ddot{\eta}_1 + 2\eta_1 - \eta_2 + g\Big\{|\xi + \eta_1| \operatorname{sgn}(\dot{\xi} + \dot{\eta}_1) - |\xi| \operatorname{sgn} \dot{\xi}$$
$$+ |2\xi + \eta_1 - \eta_2| \operatorname{sgn}(2\dot{\xi} + \dot{\eta}_1 - \dot{\eta}_2) - 2|\xi| \operatorname{sgn} \dot{\xi}\Big\} = 0 \tag{4.103}$$

$$\ddot{\eta}_2 + 2\eta_2 - \eta_1 + g\Big\{|-\xi + \eta_2| \operatorname{sgn}(-\dot{\xi} + \dot{\eta}_2) - |-\xi| \operatorname{sgn}(-\dot{\xi})$$
$$+ |-2\xi + \eta_2 - \eta_1| \operatorname{sgn}(-2\dot{\xi} + \dot{\eta}_2 - \dot{\eta}_1) - |-2\xi| \operatorname{sgn}(-2\dot{\xi})\Big\} = 0 \tag{4.104}$$

where the negative signs are "maintained" as such to closely identify with the formulation in Section IV.2. Once again, substituting $\eta_1 = \mu_1 + \mu_2$, $\eta_2 = \mu_1 - \mu_2$ and approximating the non-linear terms in

(4.103), as in (4.41)

$$\ddot{\eta}_1 + 2\eta_1 - \eta_2 + g(\rho_1 + 3\rho_2) = 0 \qquad (4.105)$$

$$\ddot{\eta}_2 + 2\eta_2 - \eta_1 + g(\rho_1 - 3\rho_2) = 0 \qquad (4.106)$$

where

$$\rho_i = |\xi + \mu_i| \operatorname{sgn}(\dot{\xi} + \dot{\mu}_i) - |\xi| \operatorname{sgn}\dot{\xi} \quad , \quad i = 1, 2 \qquad (4.107)$$

From (4.105), (4.106) and (4.107),

$$\ddot{\mu}_1 + \mu_1 + g\left\{|\xi + \mu_1| \operatorname{sgn}(\dot{\xi} + \dot{\mu}_1) - |\xi| \operatorname{sgn}\dot{\xi}\right\} = 0 \qquad (4.108)$$

$$\ddot{\mu}_2 + 3\mu_2 + 3g\left\{|\xi + \mu_2| \operatorname{sgn}(\dot{\xi} + \dot{\mu}_2) - |\xi| \operatorname{sgn}\dot{\xi}\right\} = 0 \qquad (4.109)$$

which are exactly in the form as derived in Section IV.2, in Eqn. (4.48).

Equations (4.108) and (4.109) have been analysed in Section III.4, while dealing with the stability of the periodic solution of the single degree of freedom dynamic system; accordingly $\mu_1(t)$ and $\mu_2(t)$ tend to 0 as $t \to \infty$ so that the periodic solution of $\underline{x}(t)$ in (4.101) is found to be stable.

IV. 4 Approximate Solutions by Harmonic Balance

In Section IV. 1, it has been shown that

$$\underline{x} = M^{-1/2} \underline{E}^{(j)} \xi(t) \tag{4.110}$$

is an exact periodic solution of the system of equations

$$M\ddot{\underline{x}} + S\underline{x} + gS^{*}(\underline{x})\underline{x} = \omega_j^2 M^{1/2} \underline{E}^{(j)} \sin(\omega t + \theta_j) \tag{4.111}$$

The stability of the periodic solution (4. 110) has been discussed also.

Often, however, it may be of interest to relax the restriction that the external force be in the exact modal form of any one of the eigen vectors, $\underline{E}^{(j)}$, $j=1,2,\cdot\cdot N$ of the matrix $S_1 = M^{-1/2} S M^{-1/2}$. In such a case, if not an exact solution, one might like to obtain an approximate periodic solution by fundamental harmonic balance. Such a first approximation still needs solving a set of 2N transcendental equations, which is an enormous task in itself.

It may be remarked, that an approximate solution to within fundamental harmonic balance of

$$M\ddot{\underline{x}} + S\underline{x} + gS^{*}(\underline{x})\underline{x} = M^{1/2} \underline{F}(t) \tag{4.112}$$

where

$$F_k(t) = f_k \sin(\omega t + \varphi_k) \tag{4.113}$$

is given by

$$\underline{x} = M^{-1/2} \sum_j \alpha_j \underline{E}^{(j)} A_j \sin(\omega t + \gamma_j) \tag{4.114}$$

$A_j \sin(\omega t + \gamma_j)$ is a first approximation to the solution of

$$\ddot{\psi}(t) + \omega_j^2 \{1 + g \,\mathrm{sgn}\, \psi\dot{\psi}\}\psi = \omega_j^2 \sin(\omega t + \varphi_j) \tag{4.115}$$

The α_j's and φ_j's are constants to be evaluated.

Before proceeding to substantiate the claim of the approximation to $\underline{x}$ as in (4.114), a number of useful results shall be derived.

Determination of α_j, φ_j, $j=1,2,\cdots N$

These quantities are determined in an effort to represent the external force vector as a linear combination of the eigen vectors of the matrix S_1. Since the N eigen vectors $\underline{E}^{(j)}$, $j=1,2,\cdots N$ of the matrix $S_1 = M^{-1/2} S M^{-1/2}$ are mutually orthogonal, they form a basis for an N-dimensional linear space.

Let the external force

$$F(t) = \underline{F}^* \sin \omega t + \underline{G}^* \cos \omega t \tag{4.116}$$

where $\underline{F}^*$ is the column vector $\{f_1 \cos\varphi_1,\ f_2 \cos\varphi_2, \cdots f_N \cos\varphi_N\}$ and $\underline{G}^*$ is the column vector $\{f_1 \sin\varphi_1,\ f_2 \sin\varphi_2, \cdots f_N \sin\varphi_N\}$. Then $\underline{F}^*$ can be expressed as a linear combination of the basis vectors $[\underline{E}^{(1)}, \underline{E}^{(2)}, \cdots \underline{E}^{(N)}]$ so that

$$\underline{F}^* = \sum_j a_j \omega_j^2 \underline{E}^{(j)} \tag{4.117}$$

Similarly,

$$\underline{G}^* = \sum_j b_j \omega_j^2 \underline{E}^{(j)}$$

The proportionality factor ω_j^2 is introduced in the above relations to be consistent in non-dimensionalizing the equations as was earlier done. Then (4.116) becomes

$$\underline{F}(t) = \sum_j a_j \omega_j^2 \underline{E}^{(j)} \sin \omega t + \sum_j b_j \omega_j^2 \underline{E}^{(j)} \cos \omega t$$

$$= \sum_j (a_j \sin \omega t + b_j \cos \omega t) \omega_j^2 \underline{E}^{(j)} = \sum_j \alpha_j \omega_j^2 \underline{E}^{(j)} \sin (\omega t + \varphi_j) \qquad (4.119)$$

$$\alpha_j = \sqrt{a_j^2 + b_j^2} \quad ; \quad \sin \varphi_j = \frac{b_j}{\alpha_j} \quad ; \quad \cos \varphi_j = \frac{a_j}{\alpha_j} \qquad (4.120)$$

where a_j's and b_j's are obtained from solving the systems of linear equations in (4.117) and (4.118).

Approximate solution by harmonic balance

Suppose it is desired to obtain the first approximation

$$\psi(t) = A_j \sin (\omega t + \gamma_j) \qquad (4.121)$$

to satisfy the differential equation

$$\ddot{\psi} + \omega_j^2 \{1 + g \operatorname{sgn} \psi \dot{\psi}\} \psi = \omega_j^2 \sin (\omega t + \varphi_j) \qquad (4.122)$$

Substituting (4.121) into (4.122),

$$\left(\omega_j^2 - \omega^2\right) A_j \sin (\omega t + \psi_j) + g \omega_j^2 A_j \sin (\omega t + \psi_j) \operatorname{sgn} \{\sin (\omega t + \psi_j) \cos (\omega t + \psi_j)\}$$
$$= \omega_j^2 \sin (\omega t + \varphi_j) \qquad (4.123)$$

Expanding in Fourier series both sides of (4.123) and retaining only terms corresponding to the fundamental harmonics

$$\left\{\left(\omega_j^2 - \omega^2\right) A_j \sin \chi + \frac{2g}{\pi} \omega_j^2 A_j \cos \chi + \cdots\right\} = \left\{\omega_j^2 \cos (\varphi_j - \gamma_j) \sin \chi + \omega_j^2 \cos (\varphi_j - \gamma_j) \cos \chi + \cdots\right\}$$
$$(4.124)$$

where $\chi = \omega t + \gamma_j$. Equating the coefficients of $\sin \chi$ and $\cos \chi$ respectively

$$\left(1-\frac{\omega^2}{\omega_j^2}\right)A_j = \cos(\varphi_j-\gamma_j) \quad ; \quad \frac{2g}{\pi}A_j = \sin(\varphi_j-\gamma_j) \tag{4.125}$$

or

$$A_j = \frac{1}{\left\{\left(1-\frac{\omega^2}{\omega_j^2}\right)^2+\left(\frac{2g}{\pi}\right)^2\right\}^{1/2}} \tag{4.126}$$

$$\tan\gamma_j = \frac{\left(1-\frac{\omega^2}{\omega_j^2}\right)\sin\varphi_j - \frac{2g}{\pi}\cos\varphi_j}{\left(1-\frac{\omega^2}{\omega_j^2}\right)\cos\varphi_j + \frac{2g}{\pi}\sin\varphi_j} \tag{4.127}$$

Two more results can be immediately written down following the harmonic approximation in (4.121) derived above.

(1) As already shown in Section IV. 1, the vector differential equation

$$I\ddot{\underline{y}}^{(j)} + S_1\underline{y}^{(j)} + gS_1^*(\underline{y}^{(j)})\underline{y}^{(j)} = \omega_j^2\,\underline{E}^{(j)}\sin(\omega t+\varphi_j) \tag{4.128}$$

has the exact or approximate periodic solution

$$\underline{y}^{(j)} = \underline{E}^{(j)}\psi(t) \tag{4.129}$$

depending upon whether ψ satisfies the differential equation (4.122) exactly or approximately. Since the k-th member of the N-differential equations in (4.128) is obtained by multiplying both sides of (4.122) by the factor $E_k^{(j)}$, the approximation in (4.121) implies that

$$\underline{y}^{(j)} = \underline{E}^{(j)}A_j\sin(\omega t+\psi_j) \tag{4.130}$$

satisfies (4.128) to within harmonic balance.

(2) Multiplying (4. 128) by α_j and summing over j,

$$I\sum_j \alpha_j \ddot{\underline{y}}^{(j)} + S_1 \sum_j \alpha_j \underline{y}^{(j)} + g\sum_j \alpha_j S^*(\underline{y}^{(j)})\underline{y}^{(j)} = \sum_j \alpha_j \omega_j^2 \underline{E}^{(j)} \sin(\omega t+\varphi_j) \quad (4.131)$$

Denoting

$$\sum_j \alpha_j \underline{y}^{(j)} = \underline{y}^* \quad (4.132)$$

$$I\ddot{\underline{y}}^* + S_1 \underline{y}^* + g\sum_j \alpha_j S_1^*(\underline{y}^{(j)})\underline{y}^{(j)} = \sum_j \alpha_j \omega_j^2 \underline{E}^{(j)} \sin(\omega t+\varphi_j) \quad (4.133)$$

Evidently,

$$\underline{y}^* = \sum_j \alpha_j \underline{E}^{(j)} A_j \sin(\omega t+\gamma_j) \quad (4.134)$$

$$\underline{y}^{(j)} = \underline{E}^{(j)} A_j \sin(\omega t+\gamma_j) \quad (4.135)$$

satisfy (4. 133) and (4. 128) to within fundamental harmonic balance.

<u>The first approximation for $\underline{x}$ in (4. 112)</u>

Substituting for $\underline{F}(t)$ from (4. 119) in (4. 112)

$$M\ddot{\underline{x}} + S\underline{x} + gS^*(\underline{x})\underline{x} = M^{1/2}\sum_j \alpha_j \omega_j^2 \underline{E}^{(j)} \sin(\omega t+\varphi_j) \quad (4.136)$$

for which an approximate fundamental harmonic solution is sought. Introducing $\underline{x} = M^{-1/2}\underline{y}$ and pre-multiplying (4. 136) by $M^{-1/2}$,

$$I\ddot{\underline{y}} + S_1\underline{y} + gS_1^*(\underline{y})\underline{y} = \sum_j \alpha_j \omega_j^2 \underline{E}^{(j)} \sin(\omega t+\varphi_j) \quad (4.137)$$

Consistent with (4. 114), it is claimed that $\underline{y}^*$ as in (4. 132) is the first approximation to the periodic solution of $\underline{y}$ in (4. 137).

Evidently from (4.133) and (4.137), it is sufficient to show that for the fundamental harmonics in the variable (ωt), the Fourier coefficients of the non-linear terms $S_1^*(\underline{y})\underline{y}$ when $\underline{y}=\underline{y}^*$ in (4.137) and

$$\sum_j \alpha_j S_1^*(\underline{y}^{(j)})\underline{y}^{(j)}$$

in (4.133) are identical, for all corresponding elements in the vector quantities involved.

When $\underline{y}=\underline{y}^*$, $\underline{x}=M^{-1/2}\underline{y}^*$, so that

$$(x_k - x_{k-1}) = \sum_j \left(\frac{E_k^{(j)}}{\sqrt{m_k}} - \frac{E_{k-1}^{(j)}}{\sqrt{m_{k-1}}}\right)\alpha_j A_j \sin(\omega t+\gamma_j) = R_{k,k-1}\sin(\omega t+\theta_{k,k-1}) \tag{4.138}$$

where

$$R_{k,k-1}\cos\theta_{k,k-1} = \sum_j t_{k,k-1}^{(j)}\alpha_j A_j \cos\gamma_j \tag{4.139}$$

$$R_{k,k-1}\sin\theta_{k,k-1} = \sum_j t_{k,k-1}^{(j)}\alpha_j A_j \sin\gamma_j \tag{4.140}$$

recalling

$$t_{k,k-1}^{(j)} = \frac{E_k^{(j)}}{\sqrt{m_k}} - \frac{E_{k-1}^{(j)}}{\sqrt{m_{k-1}}}$$

Then

$$\begin{aligned}\left(S_1^*(\underline{y}^*)\underline{y}^*\right)_k = &\frac{1}{\sqrt{m_k}}\Big\{-s_k R_{k,k-1}\sin(\omega t+\theta_{k,k-1}) \\ &\qquad \operatorname{sgn}[\sin 2(\omega t+\theta_{k,k-1})]\Big\} \\ &+\frac{1}{\sqrt{m_k}}\Big\{-s_{k+1}R_{k,k+1}\sin(\omega t+\theta_{k,k-1})] \\ &\qquad \operatorname{sgn}[\sin 2(\omega t+\theta_{k,k+1})]\Big\}\end{aligned} \tag{4.141}$$

$$\frac{1}{\pi}\int_0^{2\pi}\left(S_1^*(\underline{y}^*)\underline{y}^*\right)_k \sin\omega t\, d(\omega t) = \frac{1}{\pi}\sum_j \frac{1}{\sqrt{m_k}}\left\{s_k t^{(j)}_{k,k-1} + s_{k+1} t^{(j)}_{k,k+1}\right\}\alpha_j A_j \sin\gamma_j$$

$$= \frac{1}{\pi}\sum_j \left(-M^{-1/2}\, S\, M^{-1/2}\underline{E}^{(j)}\right)_k \alpha_j A_j \sin\gamma_j$$

$$= -\frac{1}{\pi}\sum_j \left(S_1\underline{E}^{(j)}\right)_k \alpha_j A_j \sin\gamma_j$$

$$= -\frac{1}{\pi}\sum_j \omega_j^2\, E_k^{(j)}\alpha_j A_j \sin\gamma_j \qquad (4.142)$$

On the other hand,

$$\sum_j \alpha_j S^*(\underline{y}^{(j)})\underline{y}^{(j)} = \sum_j \alpha_j A_j \sin(\omega t + \gamma_j)\,\mathrm{sgn}\,\{\sin 2(\omega t + \gamma_j)\} S_1 \underline{E}^{(j)} \qquad (4.143)$$

substituting for $\underline{y}^{(j)}$ from (4. 135). Therefore,

$$\left(\sum_j \alpha_j S^*(\underline{y}^{(j)})\underline{y}^{(j)}\right)_k = \sum_j \alpha_j \omega_j^2 A_j \sin(\omega t + \gamma_j) E_k^{(j)}\,\mathrm{sgn}\,\{\sin 2(\omega t + \gamma_j)\} \qquad (4.144)$$

$$\frac{1}{\pi}\int_0^{2\pi}\left(\sum_j \alpha_j S^*(\underline{y}^{(j)})\underline{y}^{(j)}\right)_k \sin\omega t\, d(\omega t) = -\frac{1}{\pi}\sum_j \omega_j^2 E_k^{(j)}\alpha_j A_j \sin\gamma_j \qquad (4.145)$$

So finally it is seen that

$$\frac{1}{\pi}\int_0^{2\pi}\left(S_1^*(\underline{y}^*)\underline{y}^*\right)_k \sin\omega t\, d(\omega t)$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\sum_j \alpha_j S^*(\underline{y}^{(j)})\underline{y}^{(j)}\right)_k \sin\omega t\, d(\omega t) \quad k = 1, 2, \cdots N \qquad (4.146)$$

Similarly the coefficients of the cos ωt term in the Fourier expansions of the above two non-linear terms are found to be identical.

This implies that $\underline{x}(t)$ as in (4.114) satisfies to within the fundamental harmonic approximation, the differential Equation (4.112). The "apparent superposition" to obtain the approximate solution has been possible strictly because of the behavior of the particular non-linearity of the problem.

Example

Consider a dynamic system with two degrees of freedom, consisting of equal masses and identical "Reid springs" just as in the examples in Section IV.3. However, suppose that the external harmonic forces have an arbitrary phase difference, 2η. The equations of motion are

$$I\ddot{\underline{x}} + S\underline{x} + gS^{*}(\underline{x})\underline{x} = \underline{F}(t) \qquad (4.147)$$

where

$$\underline{F}(t) = \begin{bmatrix} \sin(\omega t + 2\eta) \\ \sin\omega t \end{bmatrix}$$

$$= \cos\eta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(\omega t + \eta) + \sin\eta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega t + \eta) \qquad (4.148)$$

and the vector $\underline{x}$ and the matrices S, $S^{*}(\underline{x})$ are given in Equations (4.70) and (4.71).

According to (4.114), a first approximation to the periodic solution of (4.147) is given by

$$\left.\begin{aligned} x_1 &\simeq \cos\eta\,\psi_1(t) + \sin\eta\,\psi_2(t) = B_1 \sin(\omega t + \theta_1) \\ x_2 &\simeq \cos\eta\,\psi_1(t) - \sin\eta\,\psi_2(t) = B_2 \sin(\omega t + \theta_1) \end{aligned}\right\} \qquad (4.149)$$

where $\psi_1(t)$ and $\psi_2(t)$ are the periodic solutions of

$$\ddot{\psi}_1 + \{1 + g\,\mathrm{sgn}\,\psi_1\mathring{\psi}_1\}\psi_1 = \sin(\omega t + \eta)$$

$$\ddot{\psi}_2 + 3\{1+g \operatorname{sgn} \psi_2 \dot{\psi}_2\}\psi_2 = \cos(\omega t + \eta)$$

obtained by fundamental harmonic balance.

From previous results

$$\psi_1(t) \simeq A_1 \sin(\omega t + \gamma_1)$$

$$\psi_2(t) \simeq A_2 \sin(\omega t + \gamma_2)$$

$$A_1 = \left\{(1-\omega^2)^2 + \left(\frac{2g}{\pi}\right)^2\right\}^{-1/2} \quad ; \quad A_2 = \left\{(3-\omega^2)^2 + \left(\frac{6g}{\pi}\right)^2\right\}^{-1/2}$$

$$\tan\gamma_1 = \frac{(1-\omega^2)\sin\eta - \frac{2g}{\pi}\cos\eta}{(1-\omega^2)\cos\eta + \frac{2g}{\pi}\sin\eta} \quad ; \quad \tan\gamma_2 = \frac{(3-\omega^2)\cos\eta + \frac{6g}{\pi}\sin\eta}{\frac{6g}{\pi}\cos\eta - (3-\omega^2)\sin\eta}$$

Accordingly in (4. 149)

$$\left.\begin{aligned}
B_1 &= \Big\{(A_1\cos\gamma_1\cos\eta + A_2\cos\gamma_2\sin\eta)^2 \\
&\qquad + (A_1\sin\gamma_1\cos\eta + A_2\sin\gamma_2\sin\eta)^2\Big\}^{1/2} \\
B_2 &= \Big\{(A_1\cos\gamma_1\cos\eta - A_2\cos\gamma_2\sin\eta)^2 \\
&\qquad + (A_1\sin\gamma_1\cos\eta - A_2\sin\gamma_2\sin\eta)^2\Big\}^{1/2} \\
\tan\theta_1 &= \frac{A_1\sin\gamma_1\cos\eta + A_2\sin\gamma_2\sin\eta}{A_1\cos\gamma_1\cos\eta + A_2\cos\gamma_2\sin\eta} \\
\tan\theta_2 &= \frac{A_1\cos\gamma_1\cos\eta - A_2\cos\gamma_2\sin\eta}{A_1\sin\gamma_1\cos\eta - A_2\sin\gamma_2\sin\eta}
\end{aligned}\right\} \quad (4.\ 150)$$

For several values of g and η, the quantities B_1, B_2, θ_1 and θ_2 have been calculated and compared with the corresponding quantities in the fundamental Fourier components of the periodic solutions of (4. 147), obtained by numerical integration. The agreement is quite satisfactory as may be seen from the results shown in Table I for two typical cases. There are two rows of entries against each frequency in the table; the first row consists of the values calculated from (4.150) while the corresponding results obtained by numerical methods are entered in the row below.

Table Ia

$g = 0.05 \qquad \eta = \frac{\pi}{8}$

ω	B_1	θ_1	B_2	θ_2
0.625	1.528	0.437	1.532	0.245
	1.518	0.437	1.523	0.244
0.700	1.821	0.414	1.828	0.246
	1.811	0.415	1.818	0.247
0.900	4.825	0.263	4.866	0.191
	4.778	0.263	4.820	0.191
1.04	10.30	-2.402	10.46	-2.366
	10.47	-2.394	10.63	-2.359
1.50	0.830	2.912	0.957	-2.173
	0.830	2.913	0.957	-2.173
1.70	2.267	1.434	2.905	-1.997
	2.229	1.456	2.850	-1.988
2.165	0.329	-1.983	0.345	2.828
	0.330	-1.984	0.346	2.830
2.86	0.148	-2.22	0.15	3.021
	0.148	-2.22	0.15	3.021

Table Ib

$g = 0.2 \qquad \eta = \frac{3\pi}{8}$

ω	B_1	θ_1	B_2	θ_2
0.525	0.645	1.573	0.668	0.445
	0.609	1.587	0.630	0.441
0.85	1.296	1.038	1.510	0.459
	1.212	1.072	1.410	0.471
0.95	2.28	0.277	2.97	0.04
	2.12	0.412	2.70	0.14
1.0	2.50	-0.517	3.48	-0.54
	2.56	-0.360	3.45	-0.42
1.25	0.669	-2.75	1.10	-1.214
	0.666	-2.71	1.10	-1.217
1.715	2.232	1.191	2.623	-1.943
	2.194	1.341	2.586	-1.815
2.0	0.833	-0.17	0.915	-3.034
	0.855	-0.16	0.938	-3.030
2.6	0.248	-0.561	0.258	3.11
	0.249	-0.561	0.259	3.11

CHAPTER V

CONCLUSION

From earlier chapters, it follows that the "Reid Oscillator" or the piecewise-linear, non-linear model yields well-posed mathematical problems for the free and forced oscillations in dynamic systems.

In free vibrations the frequency of damped vibration is lower than in the undamped oscillator and essentially the model resembles the viscously damped system.

From the response curves in Figures 2a and 3 it is seen that during forced oscillations also the "Reid Oscillator" behaves similar to a linear system with viscous damping. On the basis of the actual computations pertaining to the exact periodic solutions described in Section III.2, it is found that the frequency at which the maximum displacement or the peak "amplitude" occurs decreases from 0.9994 to 0.99 as g increases from 0.05 to 0.2. In linear systems also, with a damping coefficient $\beta = \frac{2g}{\pi}$, the frequency at which the peak amplitude occurs, decreases from 0.9998 to 0.996 as g increases from 0.05 to 0.2.

There is one distinct feature in the present case which is not observed in forced oscillations of viscously damped linear systems. For low excitation frequency, as seen in Figure 2b, in the range $0.5 < \omega < 0.8$, the "amplitude" of vibration or maximum displacement has a greater value for larger values of g, the "non-linearity" or "damping" factor and the curves intersect each other. In viscously damped linear systems, the amplitude corresponding to a larger value of the damping ratio $\zeta = \frac{\beta}{2}$, is necessarily smaller than that for a smaller value of ζ at all frequencies.

Regarding future work, it may be pointed out that it is necessary to resolve the questions concerning the existence of ultra- and sub-harmonics. In fact for $\omega < 0.5$ in Figure 2a, it is believed from sample numerical solutions that the presence of ultra-harmonics makes it impossible to obtain periodic solutions with the symmetry in Equation (3.62). The response to random excitations would also form a major area of investigation. In certain specific materials, it may be useful to consider the presence of viscous damping or non-linear functions of the displacement also in order to derive appropriate mathematical models.

APPENDIX I

MATHEMATICAL MODELS OF MATERIALS

The most significant feature in the concept of hysteretic damping is the property that in sinusoidal motion, the energy loss per cycle due to hysteresis is independent of the frequency of oscillation. Experimental results, since the observations by Kimball and Lowell[1], have confirmed that certain materials indeed possess this property. Various attempts have been made from time to time to represent such materials in a mathematical form convenient for such applications as in vibration analysis.

Essentially there are four mathematical models based on (1) complex stiffness coefficients; (2) frequency dependent viscous damping; (3) Biot's visco-elastic model; and (4) Reid's piecewise-linear, non-linear model.

1. Complex stiffness coefficient

In this model, the restoring force is given by either

$$\text{a)}\quad F(x) = (a+ib)x$$

or

$$\text{b)}\quad F(x) = k\,e^{ig}\,x$$

where a, b, g and k are constants. Model (a) was used by Soroka[9] in his discussion of forced oscillations. Model (b) was used by Myklestad[10] and Fraeijs de Veubeke[13]. Caughey[4] and Lancaster[15] have pointed out the serious mathematical errors in these papers and as such it is impossible to draw any meaningful conclusion from their work. For instance, the transient solution presented in Myklestad's paper fails to satisfy his differential equation, indeed it satisfies the differential equation of a viscously damped system. Myklestad's solution to the forced vibration problem although correct for the type of excitation he chose, cannot be extended to other types of excitation.

2. Frequency dependent damping

In this model the restoring force is given by

$$F(x) = \frac{h}{\omega}\dot{x} + kx$$

where h and k are constants.

The interpretation of the frequency term, ω, is clear in the case of sinusoidal motion, but is far from clear for transient oscillations or complicated forcing functions. This model has been used by Mindlin[8] and Bishop[11].

The two models described above have serious mathematical and physical defects. Neither model can be simulated, even conceptually on an analog computer. The first model yields complex-valued solutions to real physical systems. The second model violates the mathematical condition for a physically realizable system.

The first successful linear model of hysteretic damping was given by Biot[27] in a paper entitled "Linear Thermodynamics and the Mechanics of Solids". At the end of this paper, he demonstrated a visco-elastic model, which at least for steady state oscillations, yields a well-posed mathematical problem.

3. Biot's visco-elastic model

In this model, the restoring force is given by

$$F(x) = k\left\{x - g\int_{t_o}^{t} Ei[-\epsilon(t-\tau)]\,\frac{dx}{d\tau}d\tau\right\}$$

where k, g and ϵ are constants and Ei is the exponential integral

$$Ei(u) = \int_{\infty}^{-u} \frac{e^{-\xi}}{\xi}\,d\xi$$

This model has been investigated by Caughey[4] who showed that it yields well-posed mathematical problems for transient and steady-state oscillations of all kinds.

4. Reid's piecewise-linear non-linear model

In this model, the restoring force is given by

$$\begin{aligned} F(x) &= k\{x + g\,|x|\,\mathrm{sgn}\,\dot{x}\} \\ &= kx\{1 + g\,\mathrm{sgn}\,x\dot{x}\} \\ &= kx + kg\left|\frac{x}{\dot{x}}\right|\dot{x} \end{aligned}$$

where k and g are constants and

$$\mathrm{sgn}\,\theta = \begin{cases} 1 \text{ if } \theta > 0 \\ 0 \text{ if } \theta = 0 \\ -1 \text{ if } \theta < 0 \end{cases}$$

Reid[12] seems to be the first to have proposed this model. In his technical note, he was preoccupied in resolving the apparent inconsistencies in the problem of free vibrations of a dynamic system with frequency dependent viscous damping; apparently, he failed to observe the far-reaching effects or the non-linear character of the model.

Hysteresis loss in a "Reid oscillator"

As just mentioned above, the restoring force is given by

$$F(x) = kx[1 + g \operatorname{sgn} x\dot{x}]$$

If the displacement x is carried through a cycle

$$x = A \sin (\omega t + \varphi)$$

then the energy loss ΔW per cycle is given by

$$\Delta W = \int_{T}^{T + 2\pi/\omega} F(x)\,\dot{x}\,dt$$

$$= kA^2 \int_{T}^{T + 2\pi/\omega} \left\{ \frac{1}{2} \sin 2(\omega t + \varphi) + g \left| \frac{1}{2} \sin 2(\omega t + \varphi) \right| \right\} d(\omega t)$$

$$= 2kgA^2$$

Thus the energy loss per cycle is proportional to the square of the amplitude, but independent of the frequency. It may be recalled that for linear viscous damping, the energy loss per cycle is proportional to the square of the amplitude and proportional to frequency also.

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